Dimensional analysis

Why ?

Check units of equations

Simplify unsolvable equations (analytical)

- Evaluate importance of each term in the equation
- Evaluate relation between physical variables

Relate model / laboratory experiments to reality

Dimensional analysis

What?

Dimensions: length, velocity, area, volume, acceleration etc.

Magnitudes / units: meter, Kg, ºC etc.

Dimensions => properties
Units => standard to quantify dimensions

Dimensional analysis => nature of the dimension

length =L; mass=M; time=T; temperature=Q

Dimensions

Quantity	SI Unit	.	Dimension
velocity	m/s	ms^{-1}	LT^{-1}
acceleration	m/s^2	ms^{-2}	LT^{-2}
force	N $kg\ m/s^2$	$kg\ ms^{-2}$	$M\ LT^{-2}$
energy (or work)	Joule J N m, $kg\ m^2/s^2$	$kg\ m^2s^{-2}$	ML^2T^{-2}
power	Watt W N m/s $kg\ m^2/s^3$	Nms^{-1} $kg\ m^2s^{-3}$	ML^2T^{-3}
pressure (or stress)	Pascal P, N/m^2, $kg/m/s^2$	Nm^{-2} $kg\ m^{-1}s^{-2}$	$ML^{-1}T^{-2}$
density	kg/m^3	$kg\ m^{-3}$	ML^{-3}
specific weight	N/m^3 $kg/m^2/s^2$	$kg\ m^{-2}s^{-2}$	$ML^{-2}T^{-2}$
relative density	a ratio no units	.	1 no dimension
viscosity	$N\ s/m^2$ kg/m s	$N\ sm^{-2}$ $kg\ m^{-1}s^{-1}$	$M\ L^{-1}T^{-1}$
surface tension	N/m $kg\ /s^2$	Nm^{-1} $kg\ s^{-2}$	MT^{-2}

Vaschy-Buckingham (Pi) theorem

“A physically meaningful equation involving a certain number, **n**, of physical variables, and these variables are expressible in terms of **r** independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of **p = n − r** <u>dimensionless variables</u> constructed from the original variables

Example: if we have n = 5 variables

Those n = 5 variables are built up from r = 3 dimensions which are:

* Length: L (m)
* Time: T (s)
* Mass: M (kg)

According to the π-theorem, the n = 5 variables can be reduced by the r = 3 dimensions to form p = n − r = 5 − 3 = **2 independent dimensionless numbers**

Vaschy-Buckingham (Pi) theorem

In mathematical terms, if we have a physically meaningful equation such as:

$$f(q_{1,} q_2, \dots, q_n) = 0$$

where the qi are the n physical variables, and they are expressed in terms of r independent physical units, then the above equation can be restated as:

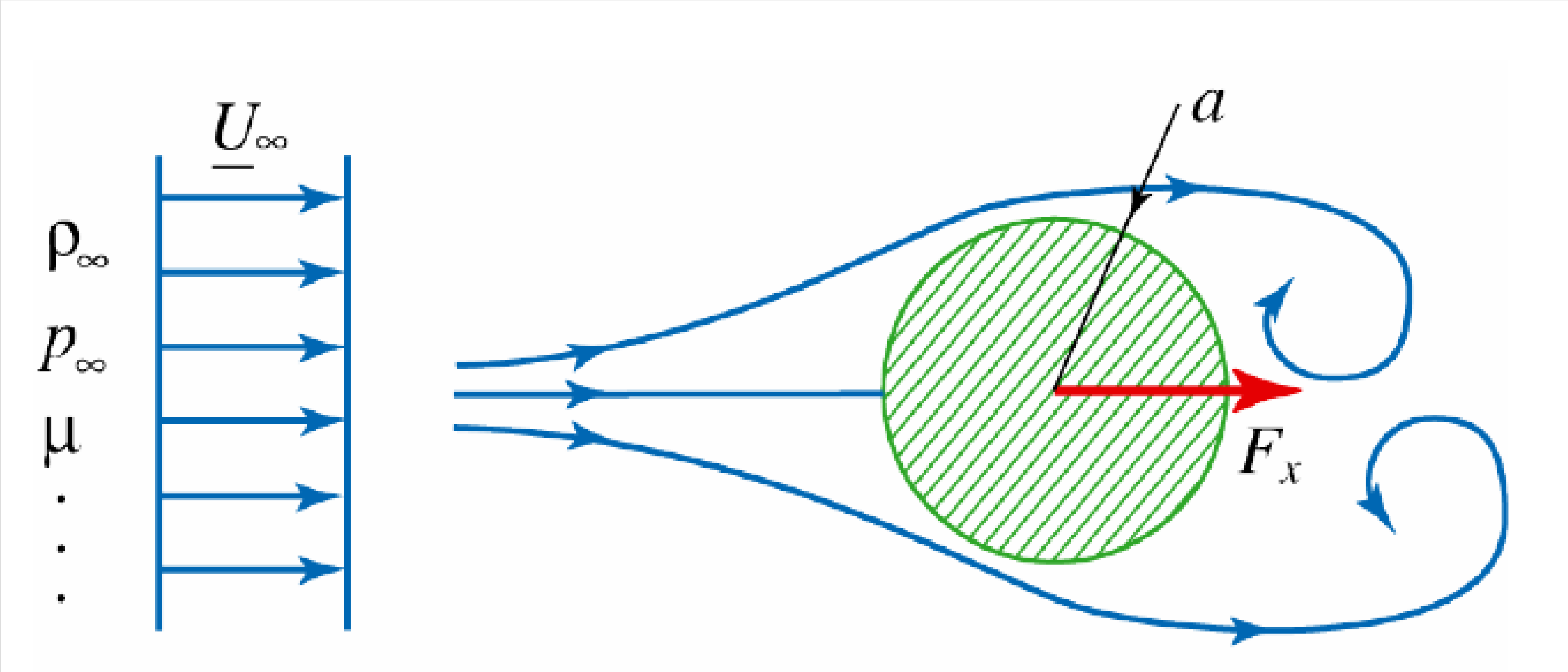
$$F(\pi_{1,} \pi_2, \dots, \pi_p) = 0$$

where the πi are dimensionless parameters constructed from the qi by p = n − r equations of the form:

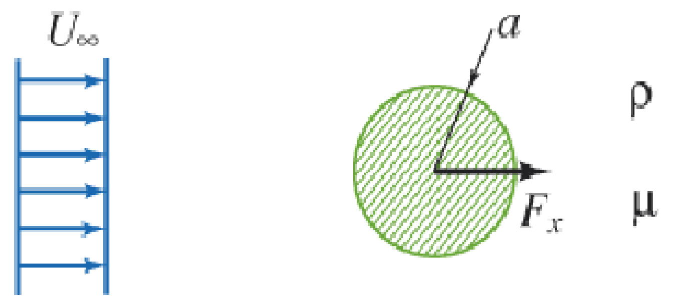
$$\pi_i = q_1^{m_1}, q_2^{m_2}, \dots, q_n^{m_n}$$

where the exponents mi are rational numbers (they can always be taken to be integers)

Flow around a cylinder



Dimensions



$$F_x = \mathcal{F}(a, U_\infty, \rho, \mu)$$

$$\begin{array}{c|ccccc|}
 & [F_x] & [a] & [U_\infty] & [\rho] & [\mu] \\
\hline
L & 0 & 1 & 1 & -3 & -1 \\
M & 1 & 0 & 0 & 1 & 1 \\
T & -2 & 0 & -1 & 0 & -1 \\
\Theta & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$N = 4 \qquad r = 3$$

Dimensionless solutions

Considering the Vaschy-Buckingham (Pi) theorem there are two possible dimensionless solutions:

$$\pi_1 = \frac{\mu}{a^{a_1} U_\infty^{a_2} \rho^{a_3}}$$

$$\pi = \frac{F_x}{a^{b_1} U_\infty^{b_2} \rho^{b_3}}$$

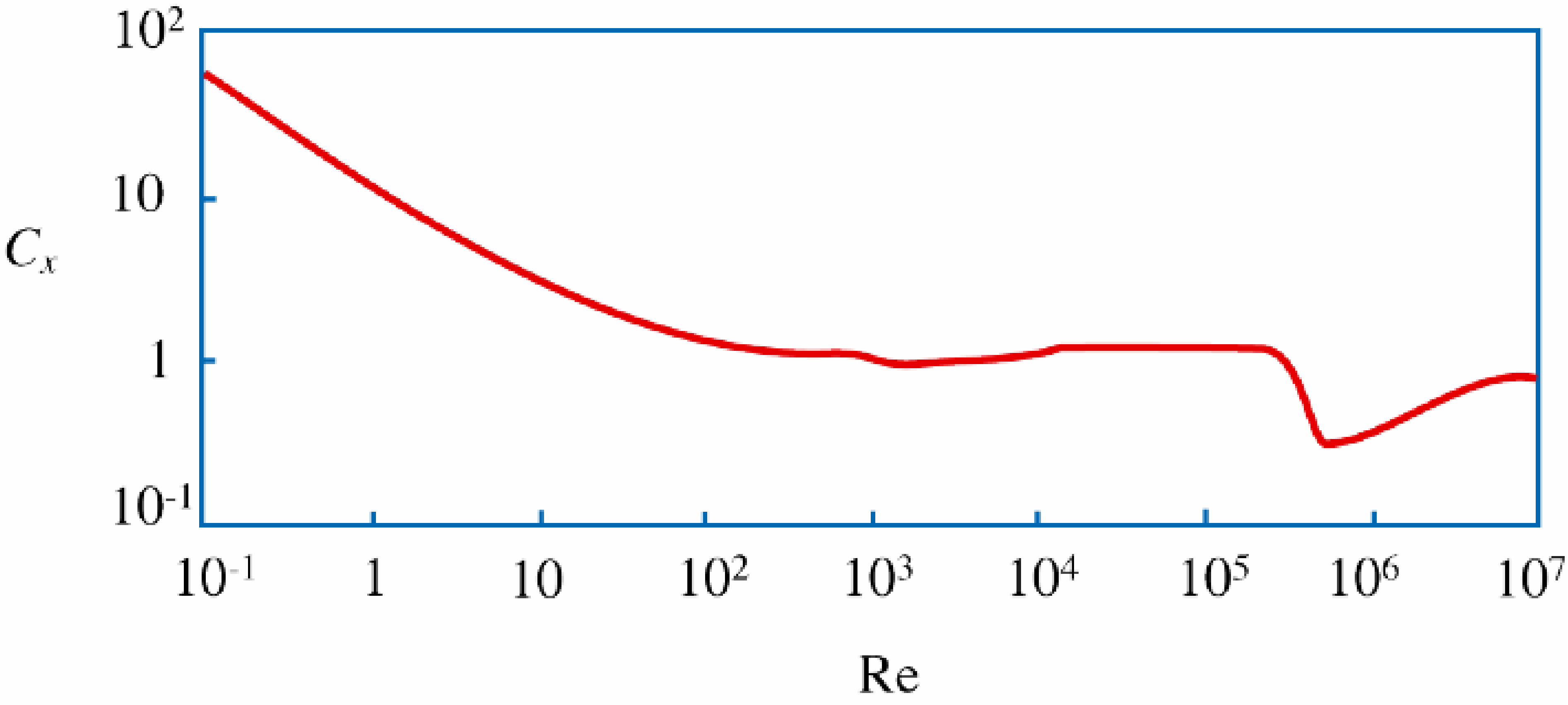
$$\pi = \overline{\mathcal{F}}(\pi_1)$$

$$\pi = \frac{F_x}{\rho U_\infty^2 a} \qquad \pi_1 = \frac{\mu}{\rho U_\infty a}$$

$$C_x = \frac{F_x}{\frac{1}{2}\rho U_\infty^2 d} \qquad \mathrm{Re} = \frac{\rho U_\infty d}{\mu}$$

$$\boxed{C_x = \overline{\mathcal{F}}(\mathrm{Re})}$$

Cylinder solutions



Re - Reynolds number

Considering Navier-Stokes equation for an incompressible fluid:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}$$

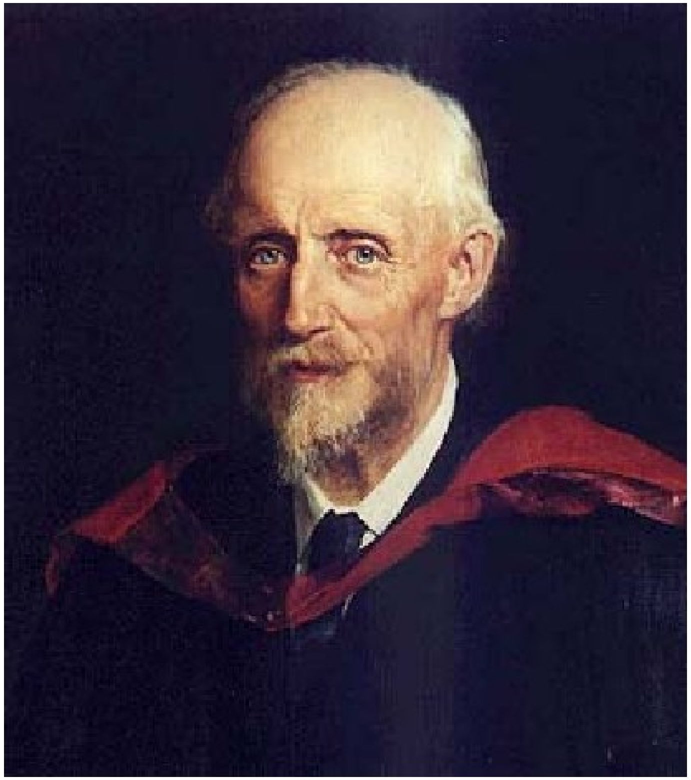
Compare inertial terms with viscous terms in steady state i.e. $\partial v/\partial t=0$

Let U = flow speed; L= size of the object in the flow; v-kinematic fluid

Inertial term: $\rho|(\mathbf{v}\cdot\nabla)\mathbf{v}| \sim \frac{\rho U^2}{L}.$ viscous term: $\eta|\nabla^2\mathbf{v}| \sim \frac{\eta U}{L^2}.$

$$\begin{aligned} \mathrm{Re} &= \frac{\text{inertial terms}}{\text{viscous terms}} \\ &= \frac{\rho U^2/L}{\eta U/L^2} \\ &= \frac{UL}{\nu}, \end{aligned}$$

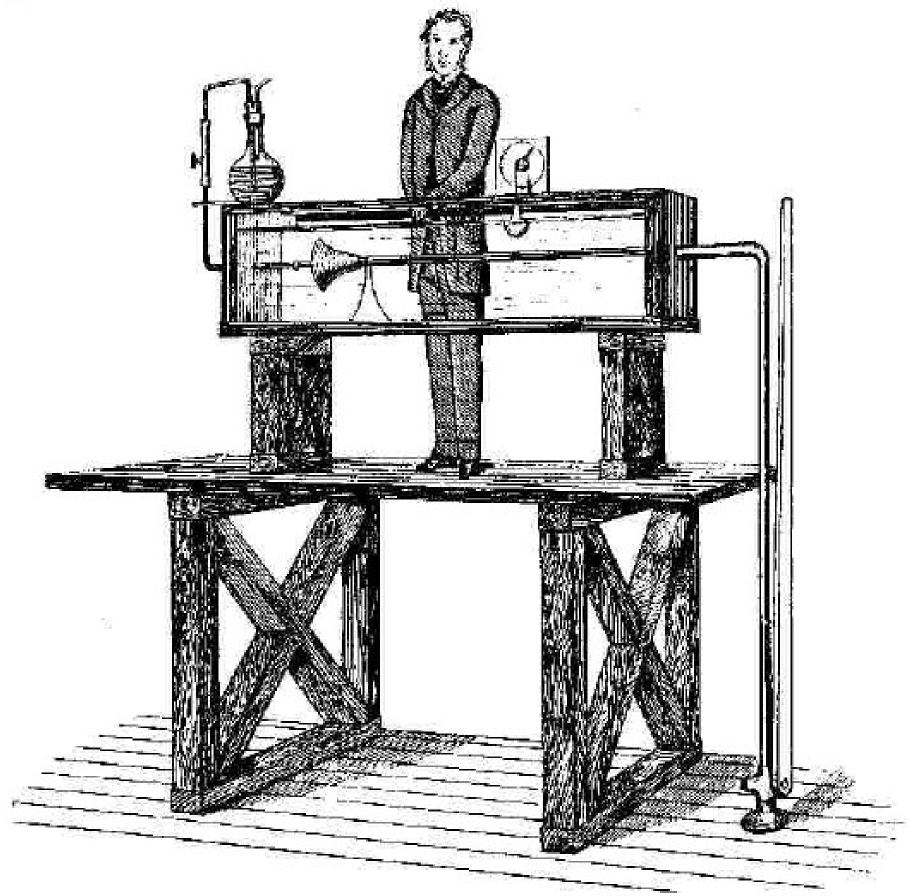
Reynolds experiments



Osborne Reynolds (1842-1912)

Reynolds experiments

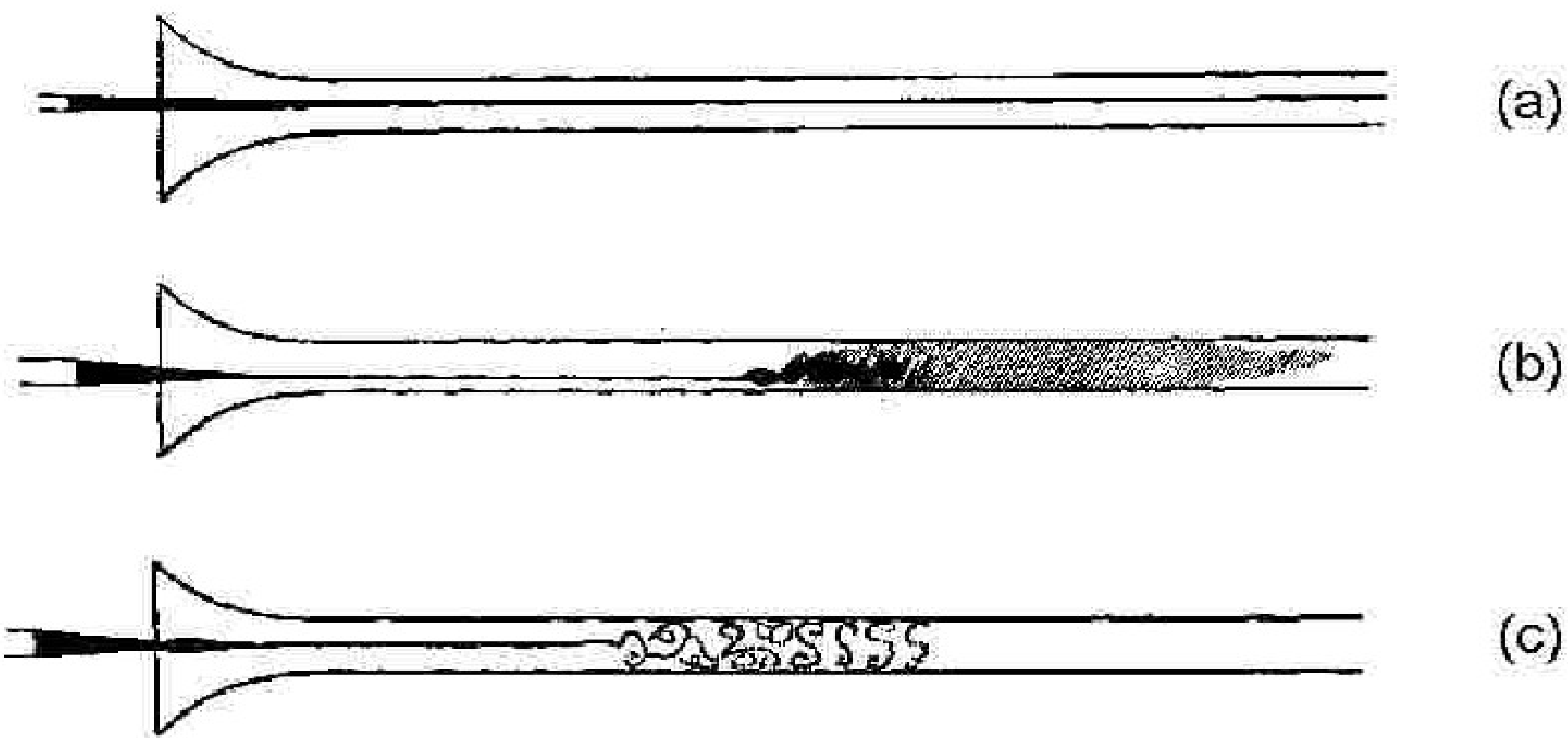
Reynolds 0. (1883) « An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels » Phil. Trans. Roy. Soc. Lond., 174, 935-982



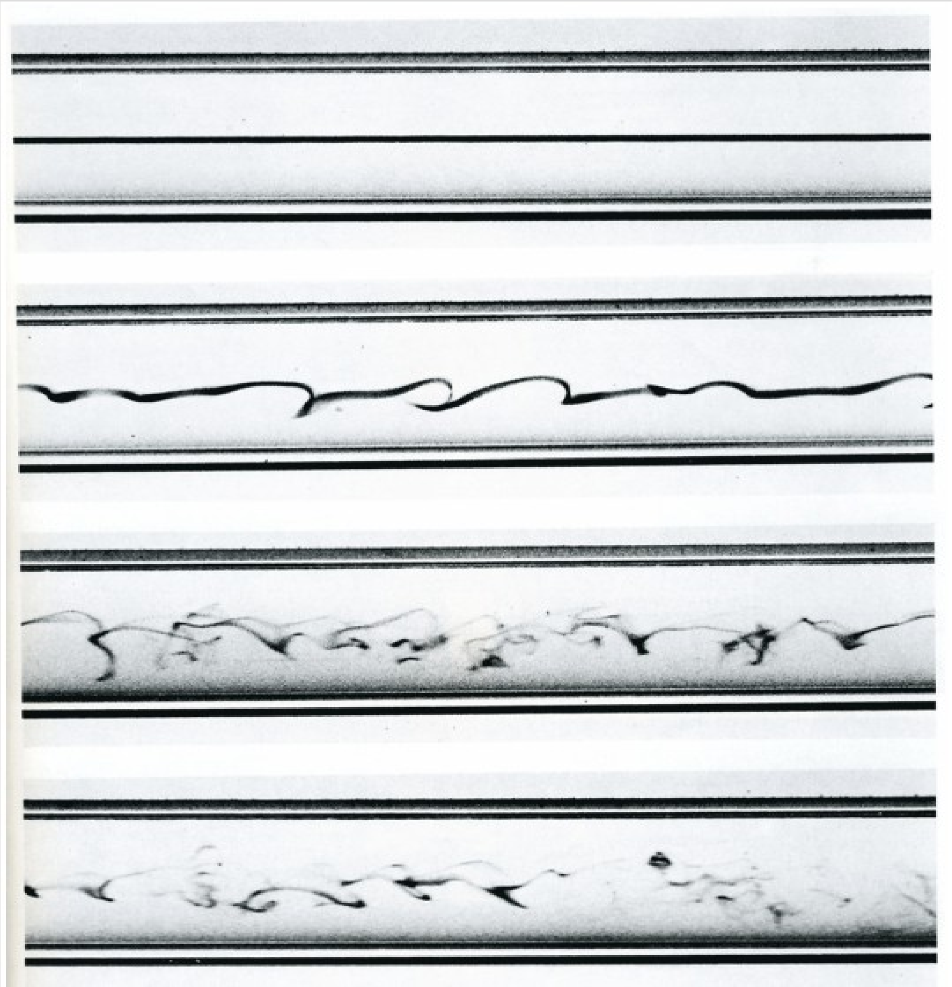
Laminar to turbulent flows

“The results of this investigation have both a practical and a philosophical aspect”

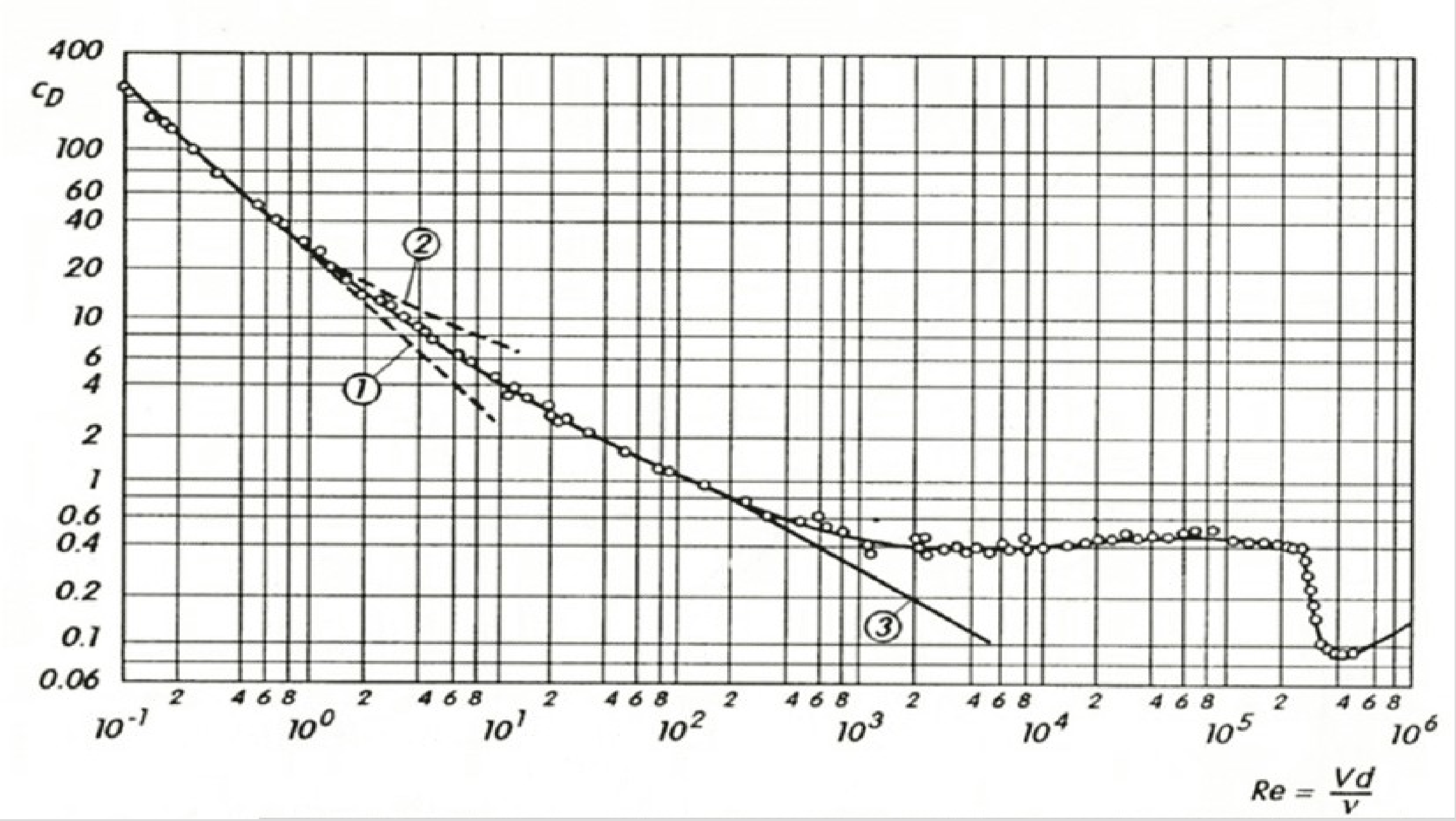
Reynolds 0. (1883)



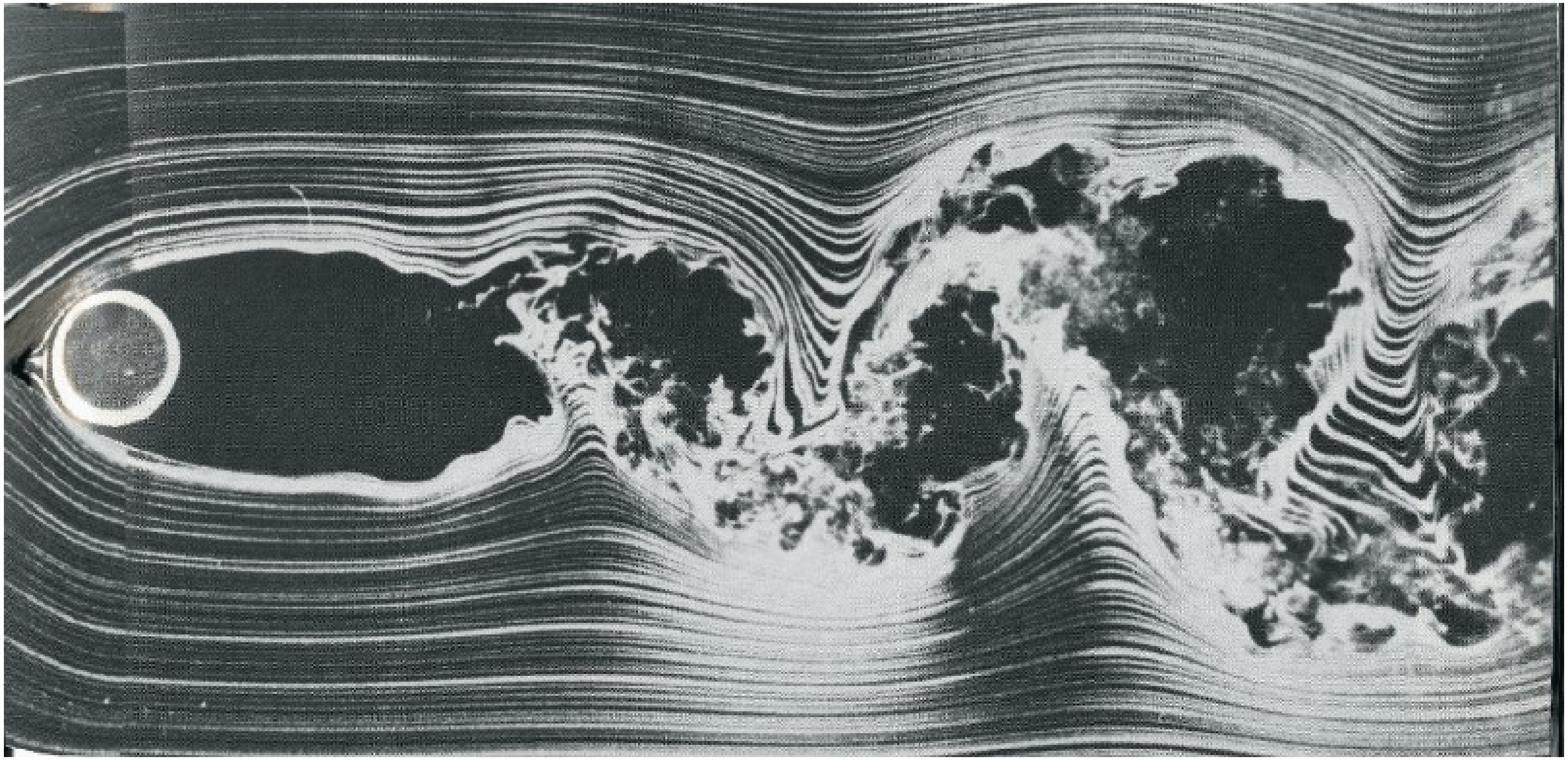
Reynolds experiments



Reynolds lab solutions



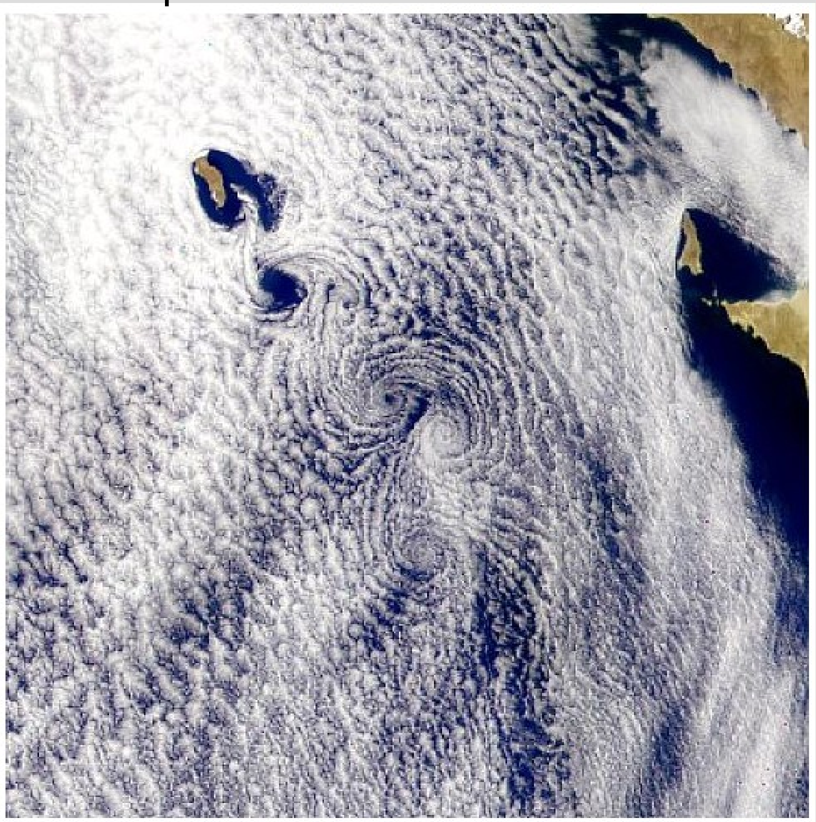
Turbulent flows in the laboratory



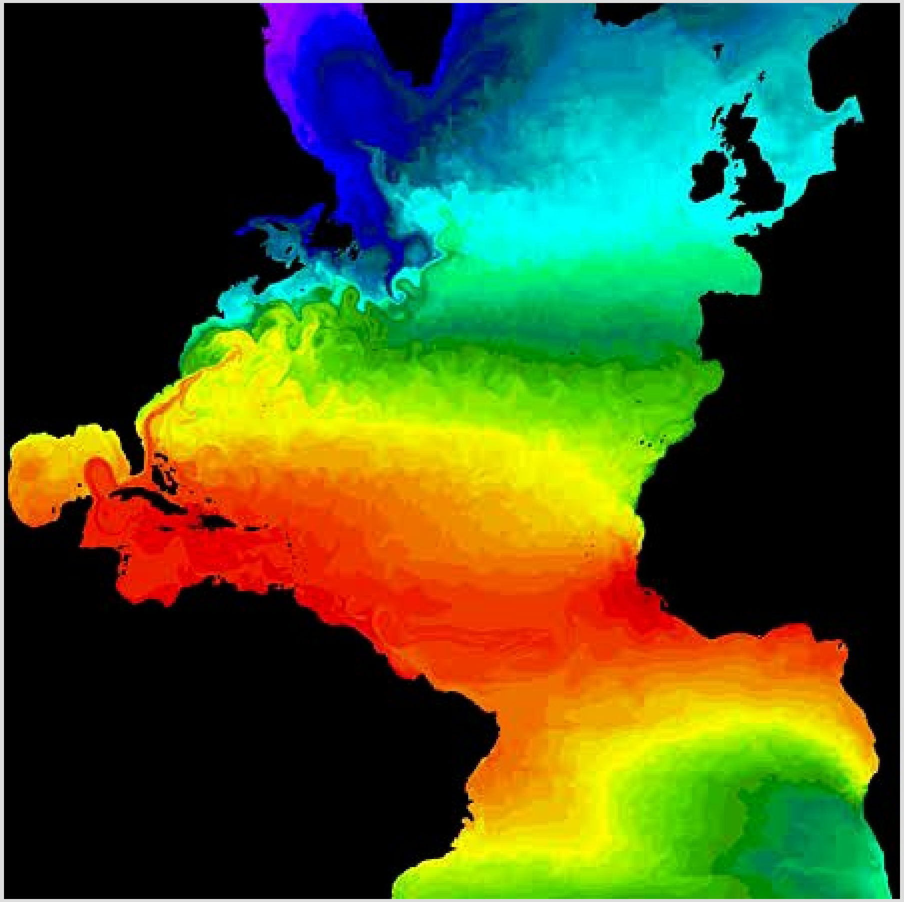
Corke & Nagib (1982)

Turbulent flows in nature

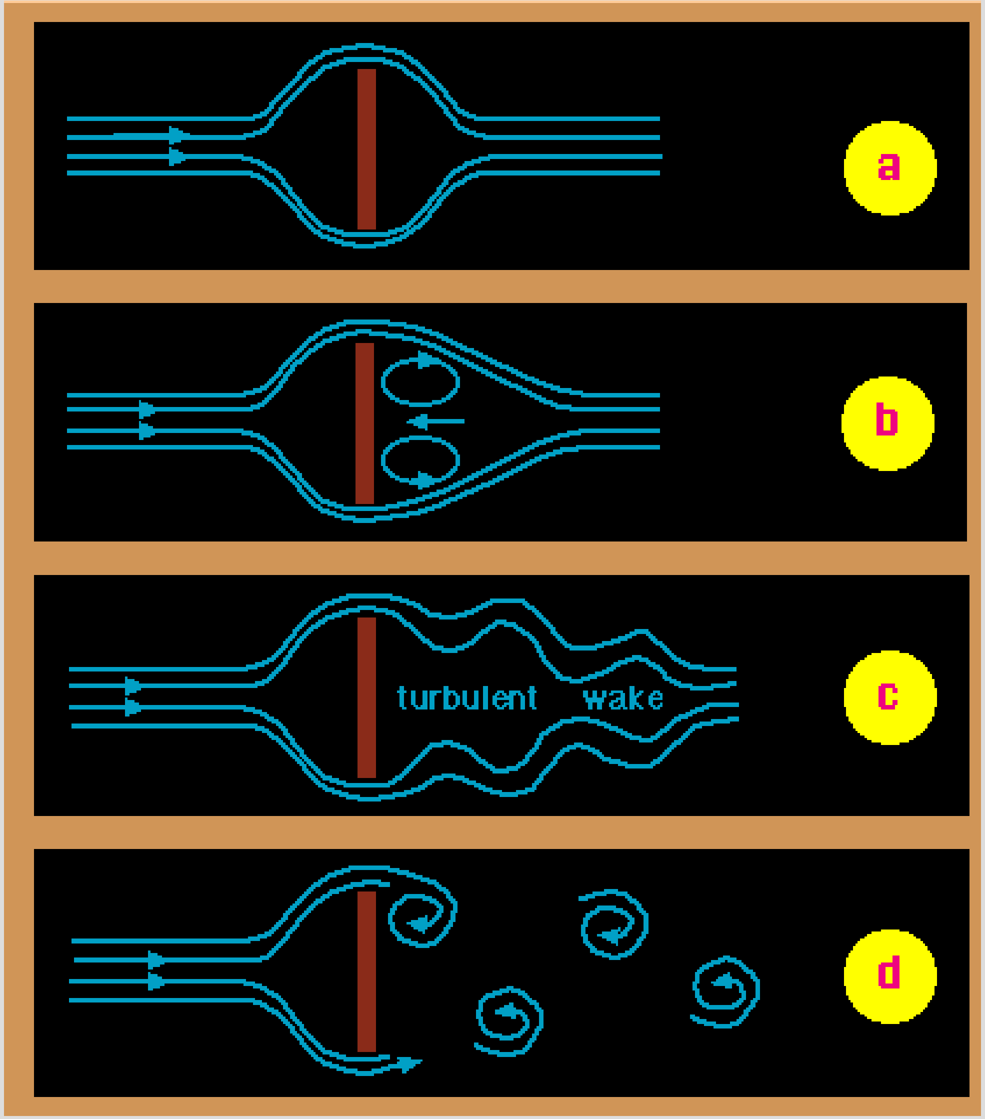
Guadalupe Island



Gulf Stream



Re & island wake regimes (lab)



No separation, laminar boundary layer
Re<0.5

Vortex pair with central return flow
2<Re<30

Wake formation with wave disturbances
40<Re<70

Von Karman vortex street
60<Re<90

Rossby number

Rossby number, named after Carl-Gustav Arvid Rossby

$$R_0 = \frac{U}{Lf}$$

Where U and L are, respectively, characteristic velocity and length scales of the phenomenon and f = 2 Ω sin φ is the Coriolis frequency, where Ω is the angular velocity of planetary rotation and φ the latitude

When the Rossby number is large (either because f is small, such as in the tropics and at lower latitudes; or because L is small, ie for small-scale motions such as flow in a bathtub; or for large speeds), the effects of planetary rotation are unimportant and can be neglected. When the Rossby number is small, then the effects of planetary rotation are large and the net acceleration is comparably small allowing the use of the geostrophic approximation

Ek – Ekman number

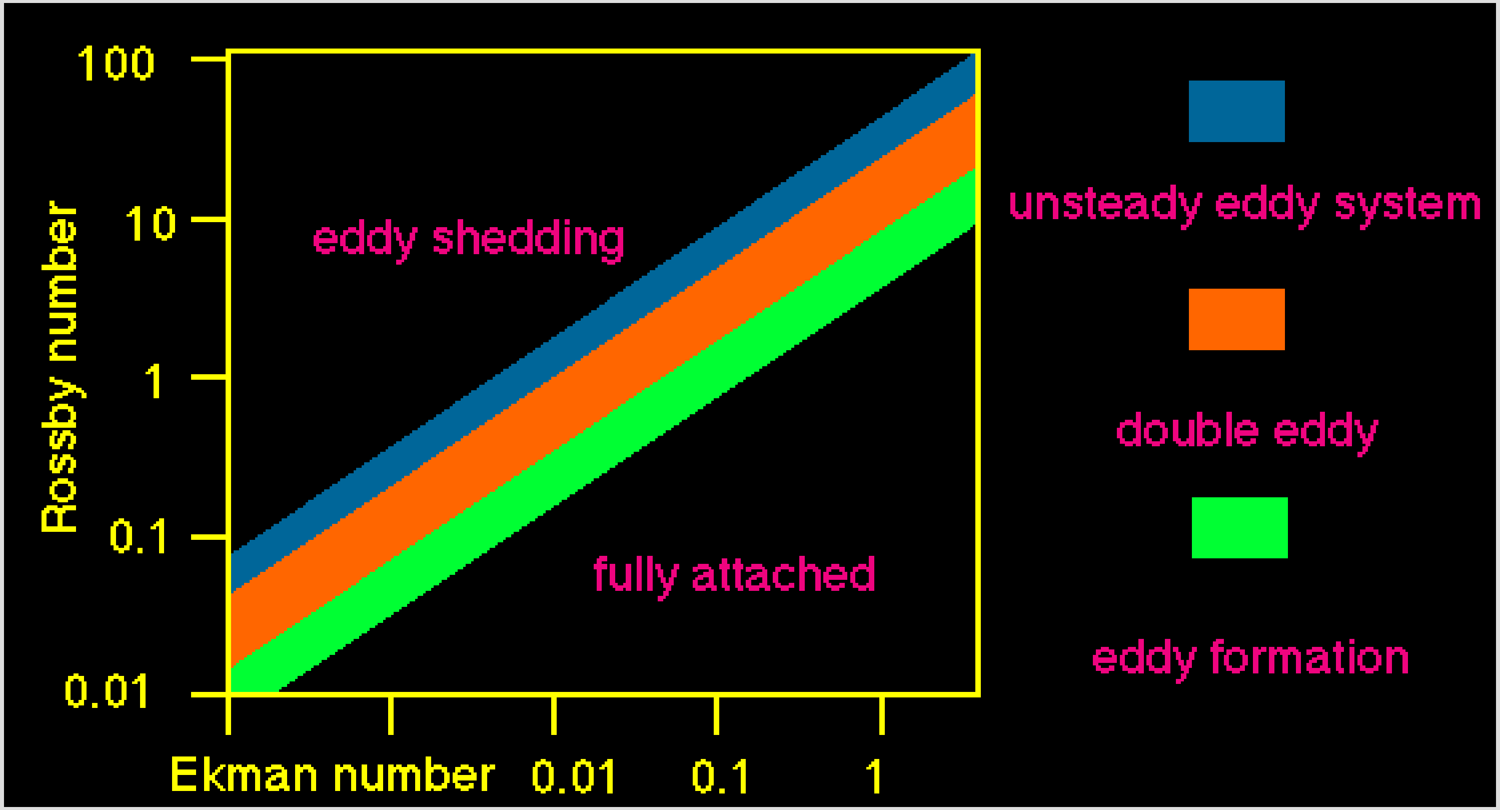
Named after V. Walfrid Ekman

Ratio of viscous forces to the Coriolis forces

$$Ek = \frac{\nu}{fL^2}$$

When the Ekman number is small, frictional effects are negligible

Ro / Ek wake regimes

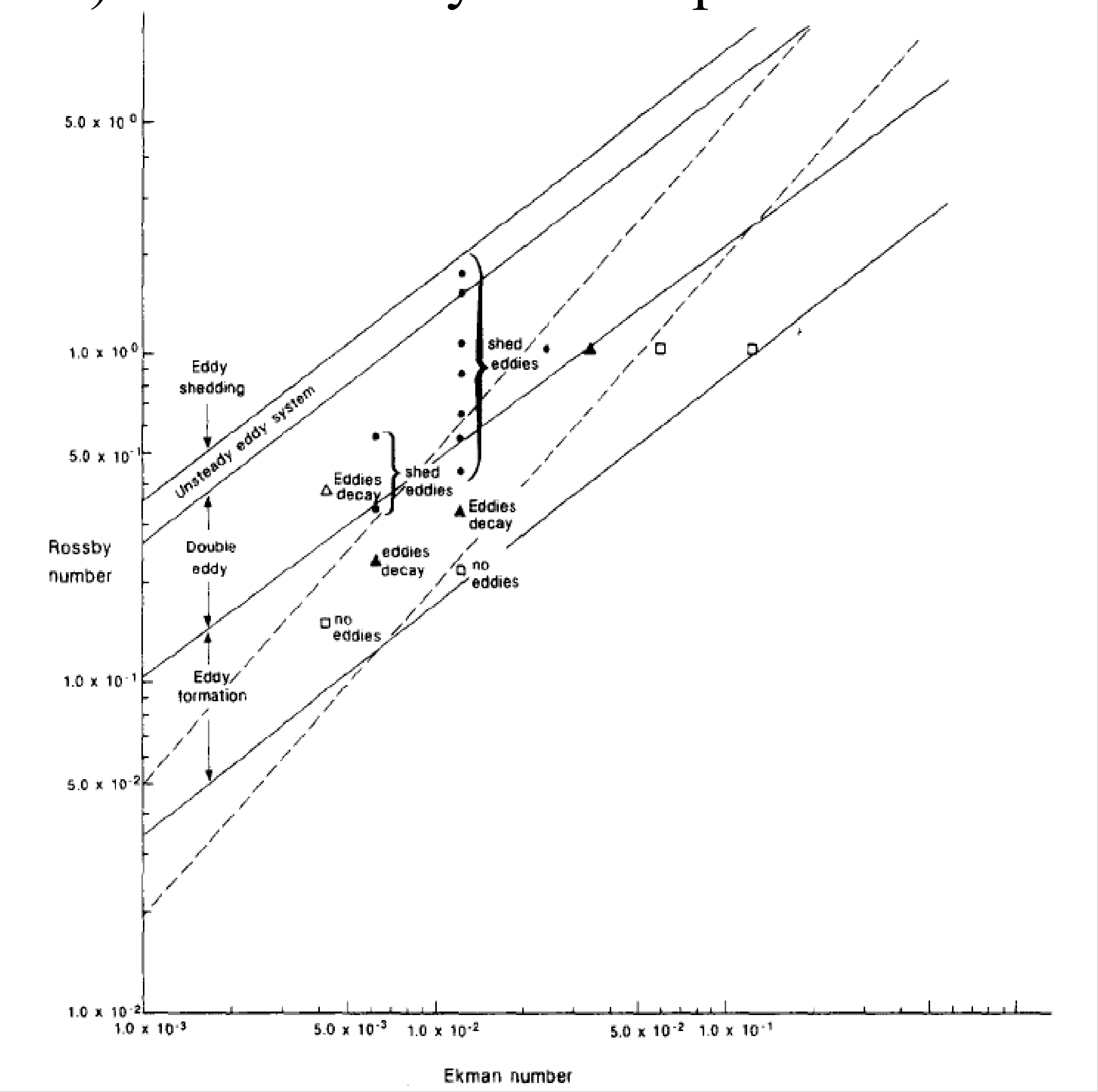


Pattiarachi, C., A. James and M. Collins (1986) Island wakes and headland eddies: a comparison between remotely sensed data and laboratory experiments. Journal of Geophysical Research 92, 783 - 794.

Ro / Ek wake regimes

Baines & Davies (1980) extended Reynolds experiments to rotating flows

$$R_e = \frac{Ro}{Ek}$$



Strouhal number

Named after Vincenc Strouhal, a Czech specialized in experimental physics

$$St = \frac{fL}{U}$$

Where f – eddy frequency; L – length scale; U – fluid velocity

St / Re wake regimes