Modeling concepts

What are numerical models ?

Set of Equations

↓

Numerical algorithms

↓

Computer codes

Numerical methods

Why?

Integrals or PDEs rarely have analytic solutions

Computers can only solve algebraic equations

Hydrodynamics are equations are based on mass, momentum and energy PDEs are necessary to describe **“rates of change”** which describes the conservation principle. In general all pysical processes can be describes with PDEs.

One way of PDEs using computers is by numerical discretisation techniques i.e. transform each differential term into an approximate algebraic equation:

Numerical methods

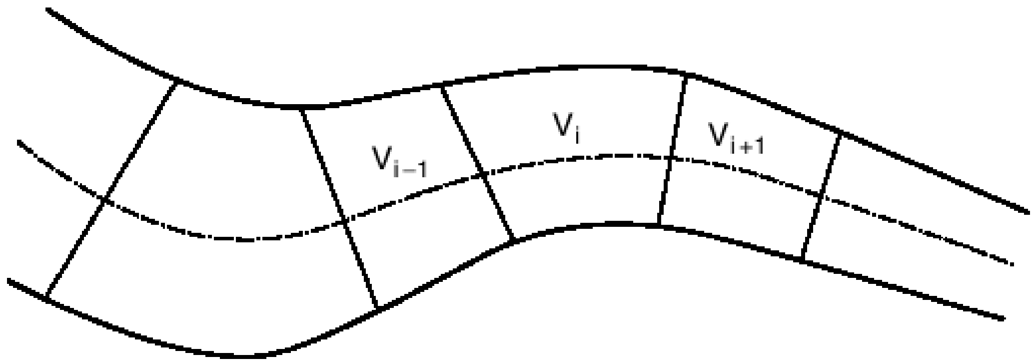
Transport equation:

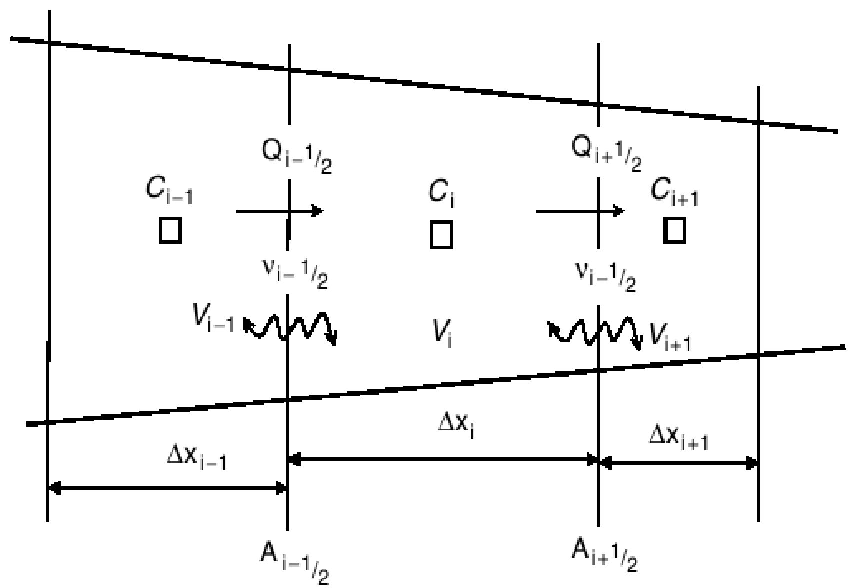
PDE $$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \nu\frac{\partial^2 C}{\partial x^2}$$

Finite-differences discretisation:

$$\frac{C_i^{t+\Delta t} - C_i^{t}}{\Delta t} = \left(U\frac{C_{i-1/2} - C_{i+1/2}}{\Delta x}\right)^{t=t^*} + \nu\left(\frac{C_{i-1} - 2C_i + C_{i+1}}{\Delta x^2}\right)^{t=t^*}$$

Spatial discretisation





Numerical methods

There are several methods for PDE discretization in CFD

- Finite differences
  - Explicit
    - central dif.
    - upstream
  - Implicit
  - Crank-Nicolson
- Finite volume
- Finite element

There are also various discretisation schemes for specific terms e.g. Convection term discretised using upwind QUICK SOU etc.

So which method ? → **stability**

1D model

Considering a no diffusion case of the transport equation:

$$\frac{C_i^{t+\Delta t}-C_i^t}{\Delta t} = \left(U\frac{C_{i-1/2}-C_{i+1/2}}{\Delta x}\right)^{t=t^*} + \nu\left(\frac{C_{i-1}-2C_i+C_{i+1}}{\Delta x^2}\right)^{t=t^*}$$

Central explicit differences $$C_i^{t+\Delta t} = \frac{1}{2}\frac{U\Delta t}{\Delta x}C_{i-1}^t + C_i^t - \frac{1}{2}\frac{U\Delta t}{\Delta x}C_{i-1}^t$$

Where: $\frac{U\Delta t}{\Delta x} = C_r$ Courant number: time / grid size

So is this this model stable with Cr=1 or Cr =2 ?

Analise model solution ? Is it real? Physically possible?

1D model – explicit CD

Time Step	Grid point i-3	i-2	i-1	I	I+1	I+2	I+3	SUM
0	**0**	**0**	**0**	**1**	**0**	**0**	**0**	1
1	**0**	0.00	-0.50	1.00	0.50	0.00	**0**	1
2	**0**	0.25	-1.00	0.50	1.00	0.25	**0**	1
3	**0**	0.75	-1.13	-0.50	1.13	0.75	**0**	1
4	**0**	1.31	-0.50	-1.63	0.50	1.31	**0**	1
5	**0**	1.56	0.97	-2.13	-0.97	1.56	**0**	1
6	**0**	1.08	2.81	-1.16	-2.81	1.08	**0**	1
7	**0**	-0.33	3.93	1.66	-3.93	-0.33	**0**	1
8	**0**	-2.29	2.94	5.59	-2.94	-2.29	**0**	1
9	**0**	-3.76	-1.00	8.52	1.00	-3.76	**0**	1
10	**0**	-3.26	-7.14	7.52	7.14	-3.26	**0**	1
11	**0**	0.31	-12.54	0.38	12.54	0.31	**0**	1

Stability explicit CD

Why ? because water entering a channel should move forward in time

So after a long enough time the entire channel should have concentration equal to 0

A model is said unstable when errors amplify;

Error growth higher than Cr

"The influence of point on its neighbors through advection or diffusion can not be negative"

Stability explicit CD

Advection　　　　Diffusion

$$\frac{C_i^{t+\Delta t} - C_i^t}{\Delta t} = \left(U \frac{C_{i-1/2} - C_{i+1/2}}{\Delta x} \right)^{t=t^*} + \nu \left(\frac{C_{i-1} - 2C_i + C_{i+1}}{\Delta x^2} \right)^{t=t^*}$$

$$Re = \frac{U \Delta x}{\nu} \leq 2 \quad \longleftrightarrow \quad d = \frac{\nu \Delta x}{\Delta x^2} \leq \frac{1}{2}$$

$$Re = \infty \qquad d \leq \frac{1}{2}$$

So what happens if this criteria is meet ?

Stability explicit CD

i-1	I	I+1	I+2	I+3	SUM	Courant	Diffusion #	
	0	**1**	**0**	**0**	**0**	1	1	0.5
	0.00	0.00	1.00	0.00	**0**	1	1	0.5
	0.00	0.00	0.00	1.00	**0**	1	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5
	0.00	0.00	0.00	0.00	**0**	0	1	0.5

Explicit upstream

$$C_i^{t+\Delta t} = \frac{U\Delta t}{\Delta x} C_{i-1}^t + \left(1 - \frac{U\Delta t}{\Delta x}\right) C_i^t \quad (U > 0)$$

Time Step	Grid point							
	i-3	i-2	i-1	I	I+1	I+2	I+3	SUM
0	**0**	**0**	**0**	**1**	**0**	**0**	**0**	1
1	**0**	0.00	0.00	0.00	1.00	0.00	**0**	1
2	**0**	0.00	0.00	0.00	0.00	1.00	**0**	1
3	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
4	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
5	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
6	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
7	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
8	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
9	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
10	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0
11	**0**	0.00	0.00	0.00	0.00	0.00	**0**	0

Stability: $C_r \leqslant 1$ $\quad (C_r + 2\mathrm{d}) \leqslant 1$

$C_r = 0.5$ → numerical diffusion

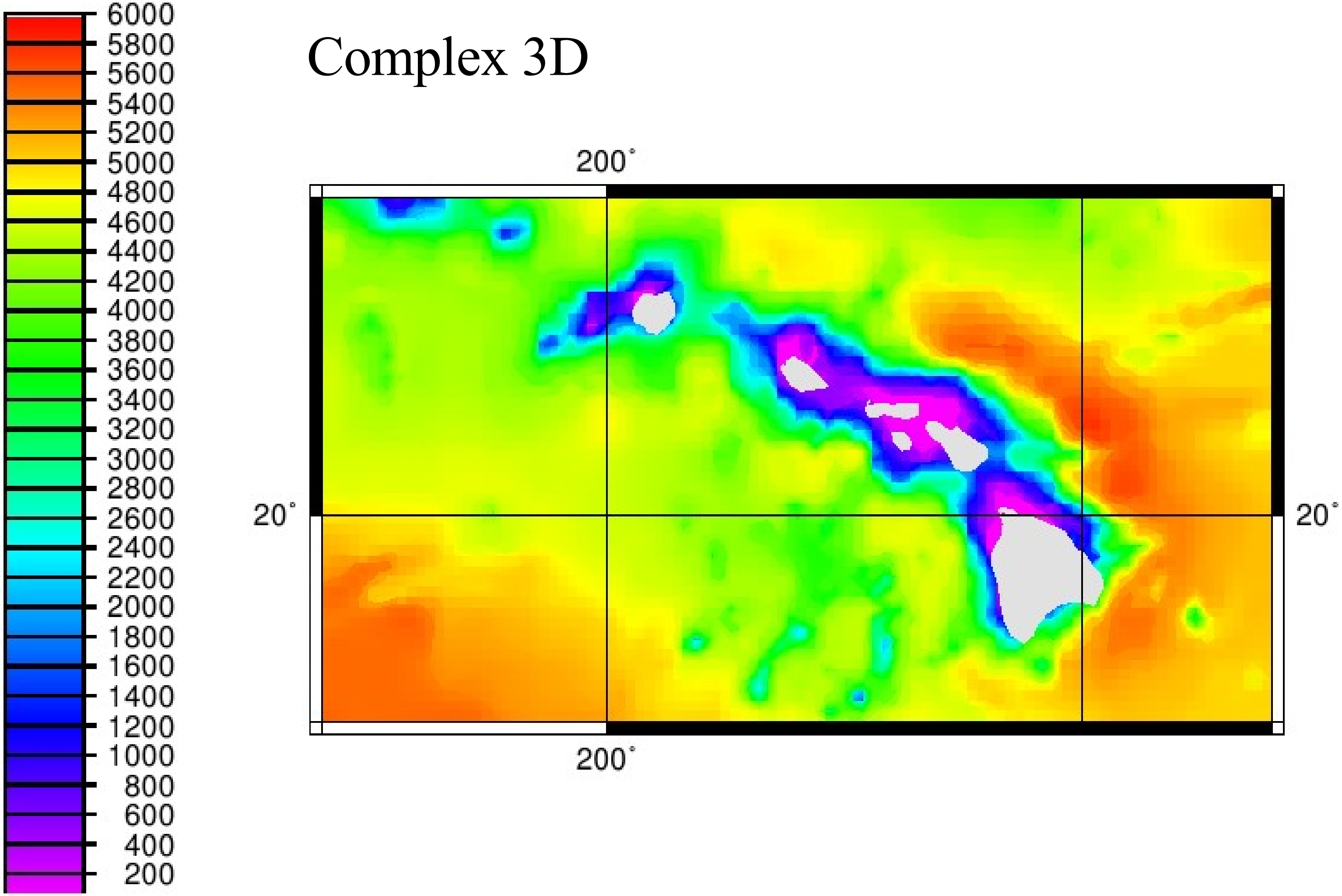
Numerical methods

In general, numerical methods in the model have to:

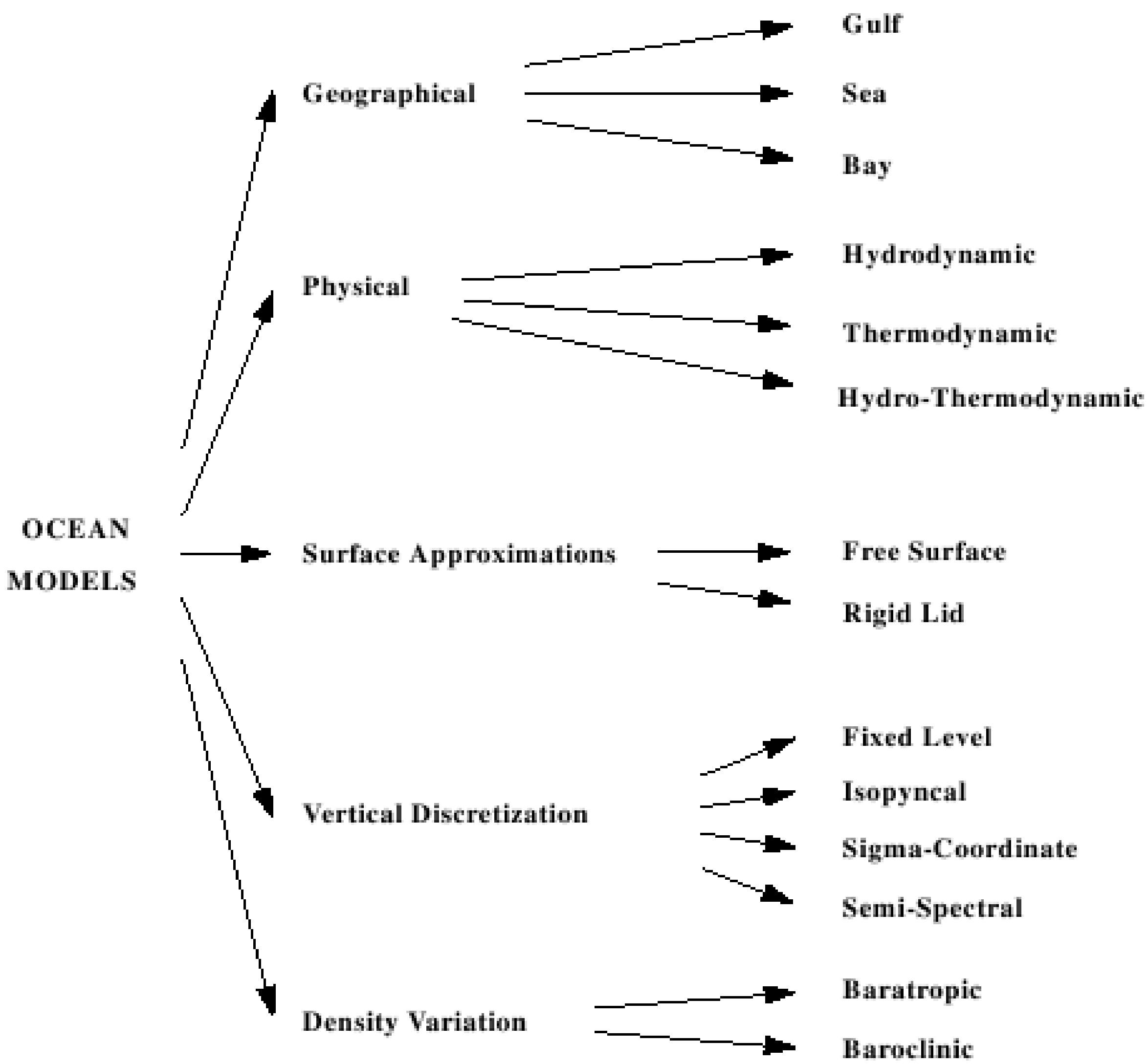
- Maintain mass conservation
- Guarantee stability
- Numerical diffusion must be smaller than physical diffusion

Ocean domains

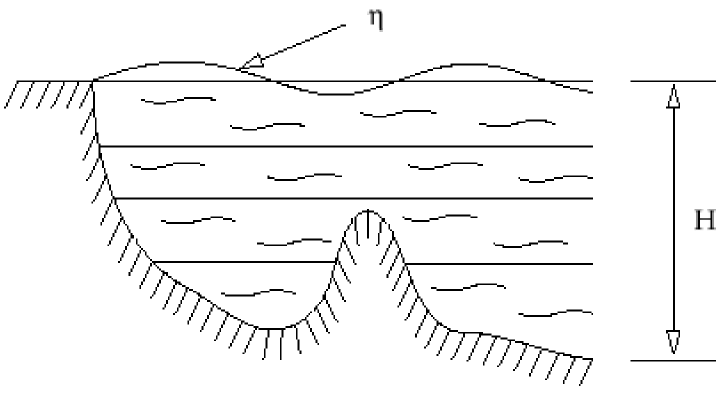
Complex 3D



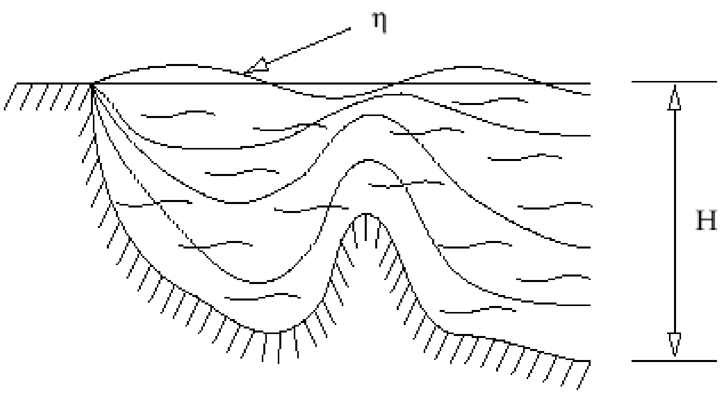
Ocean circulation models



Ocean circulation models



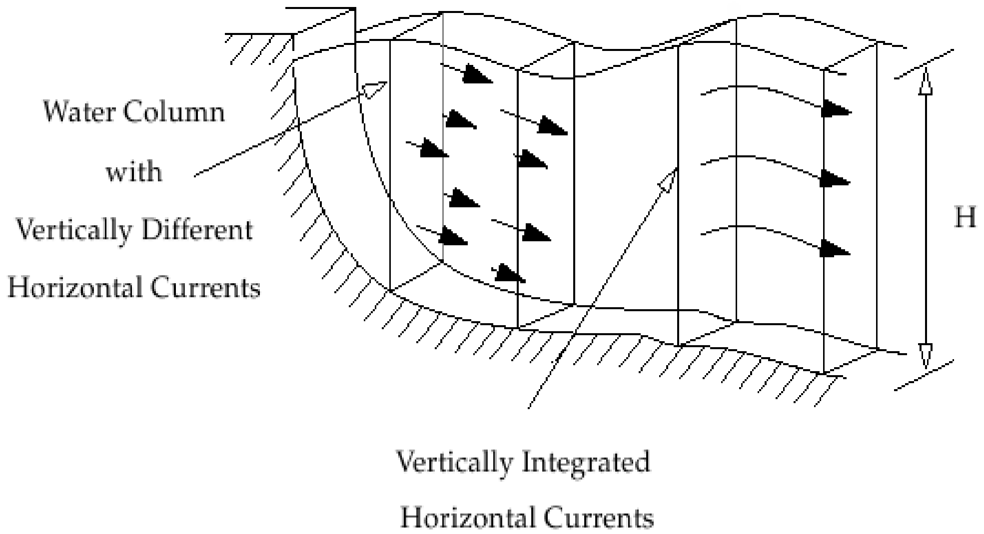
z-Coordinate Model

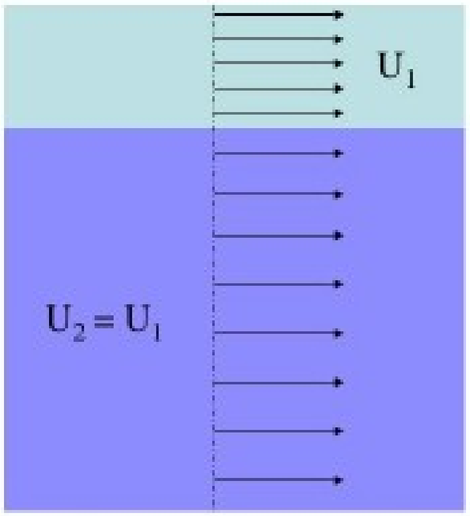


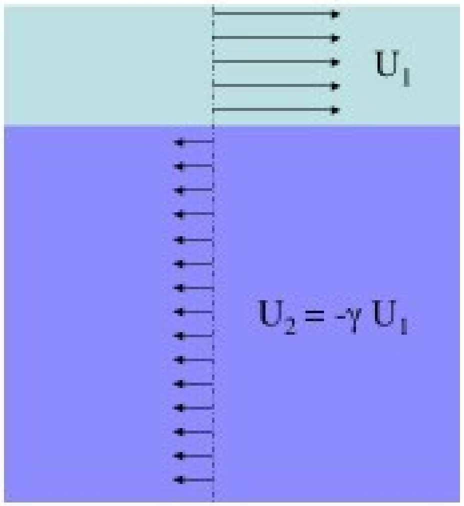
σ-Coordinate Model

$$\sigma = \frac{z}{D}$$

Barotropic / Baroclinic models







Hydrostatic approximation

$$\frac{dp}{dz} = \rho g$$

Barotropic model

Physical principles for the ocean interior:

$$\frac{dU}{dt} = +fV - gH\left(\frac{d\eta}{dx}\right) + (\tau_{\mathrm{w}} - \tau_{\mathrm{b}})_x + A\nabla^2 U - \frac{d}{dx}(UU/H) - \frac{d}{dy}(UV/H),$$

$$\frac{dV}{dt} = -fU - gH\left(\frac{d\eta}{dy}\right) + (\tau_{\mathrm{w}} - \tau_{\mathrm{b}})_y + A\nabla^2 V - \frac{d}{dx}(UV/H) - \frac{d}{dy}(VV/H),$$

$$\frac{d\eta}{dt} = -\left(\frac{dU}{dx} + \frac{dV}{dy}\right)$$

Barotropic model

$\vec{U} = (U, V)$	—mass transports in the x- and y-directions, respectively
$\vec{\tau}_{\mathrm{w}} = [(\tau_{\mathrm{w}})_x, (\tau_{\mathrm{w}})_y]$	—wind stress components
$\vec{\tau}_{\mathrm{b}} = [(\tau_{\mathrm{b}})_x, (\tau_{\mathrm{b}})_y]$	—bottom stress components
η	—free surface elevation
$f = 2\Omega \sin\phi$	—Coriolis parameter for latitude ϕ; Ω is the angular rotation rate of the Earth ($7 \times 10^{-5}\mathrm{s}^{-1}$)
$H(x, y)$	—topography (bottom depth)
g	—acceleration due to gravity
∇^2	—Laplacian operator in horizontal coordinates x, y
A	—coefficient of lateral friction
Δx	—grid spacing in the longitudinal direction
Δy	—grid spacing in the latitudinal direction
Δt	—time step

Bondary conditions

$U = V = 0$ No tangential velocities ⟶ friction

$$\frac{d\eta}{dx} = \frac{1}{gH}\left[(\tau_{\mathrm{w}} - \tau_{\mathrm{b}})_x \left(\frac{d^2U}{dx^2}\right)\right] \quad \text{at } x = 0, L_x$$

$$\frac{d\eta}{dy} = \frac{1}{gH}\left[(\tau_{\mathrm{w}} - \tau_{\mathrm{b}})_y \left(\frac{d^2V}{dy^2}\right)\right] \quad \text{at } y = 0, L_y.$$

Friction

Bottom

$$\tau_{\mathrm{b}_x} = C_{\mathrm{d}} \left| \vec{U} \right| U$$

$$\tau_{\mathrm{b}_y} = C_{\mathrm{d}} \left| \vec{U} \right| V.$$

Lateral (horizontal)

$$\frac{d^2 \eta}{dt^2} = gH\nabla^2 \eta + g\Delta H \cdot \nabla \eta .$$

Explicit time discretization

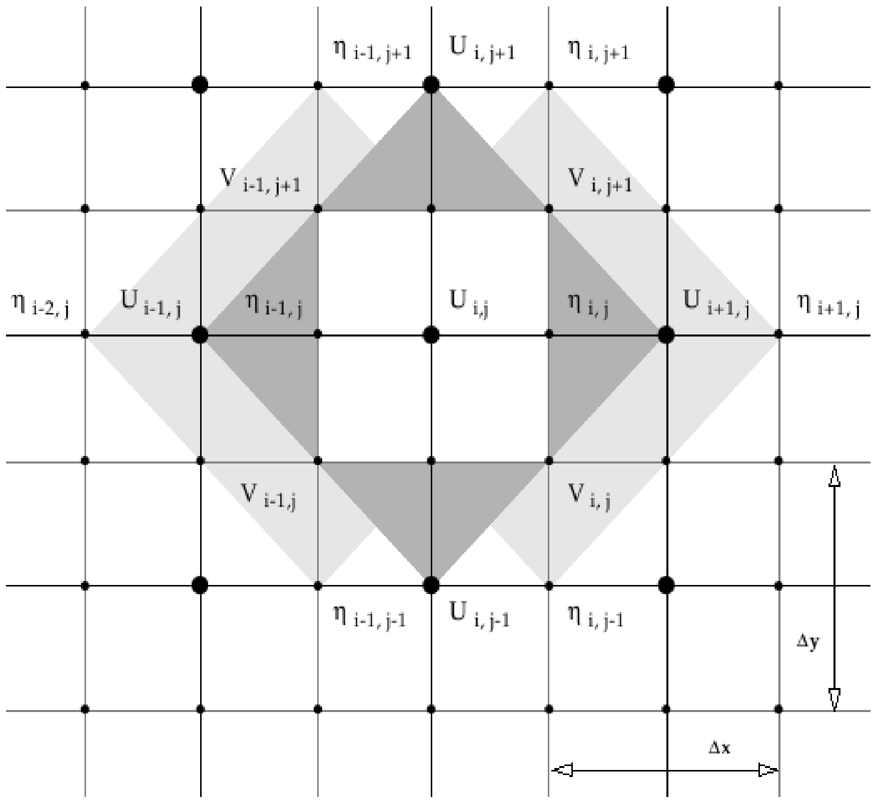
$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n-1}}{\Delta t}$$

$$= fV_{i,j}^{n} - \frac{gH}{\Delta x}(\eta_{i,j}^{n} - \eta_{i-1,j}^{n})$$

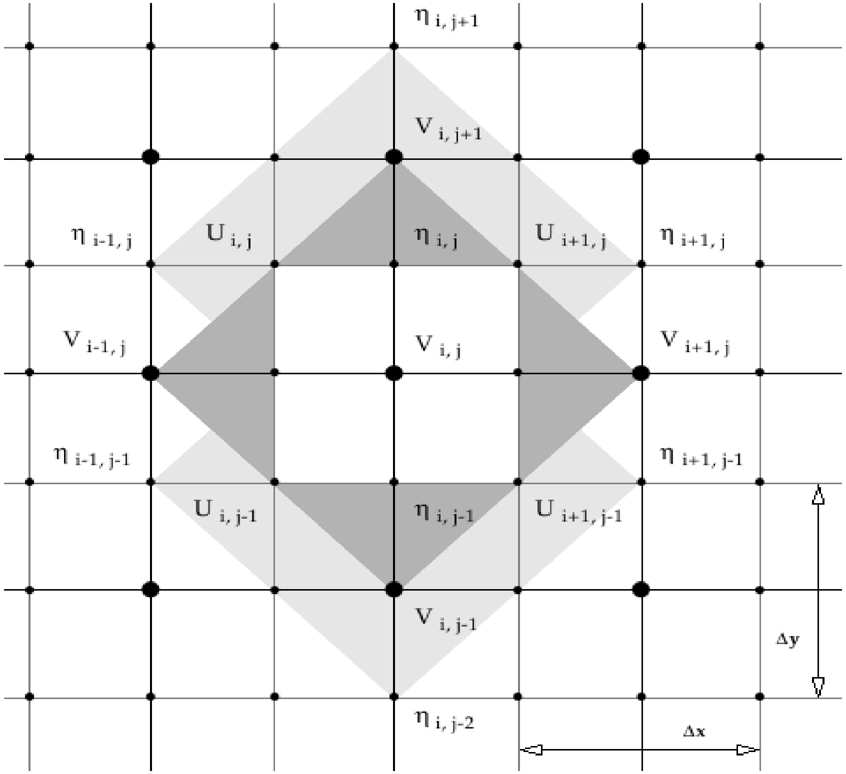
$$+ \quad \frac{A}{\Delta x^2}(U_{i+1,j}^{n-1} + U_{i-1,j}^{n-1}$$

$$+ U_{i,j+1}^{n-1} + U_{i,j-1}^{n-1} - 4U_{i,j}^{n-1})$$

$$+ \quad (\tau_x)_{i,j}^{n} - C_{\mathrm{d}} U_{i,j}^{n-1}$$



Explicit time discretization



$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n-1}}{\Delta t}$$

$$= fU_{i,j}^{n} - \frac{gH}{\Delta y}(\eta_{i,j}^{n} - \eta_{i,j-1}^{n})$$

$$+ \quad \frac{A}{\Delta y^2}(V_{i+1,j}^{n-1} + V_{i-1,j}^{n-1}$$

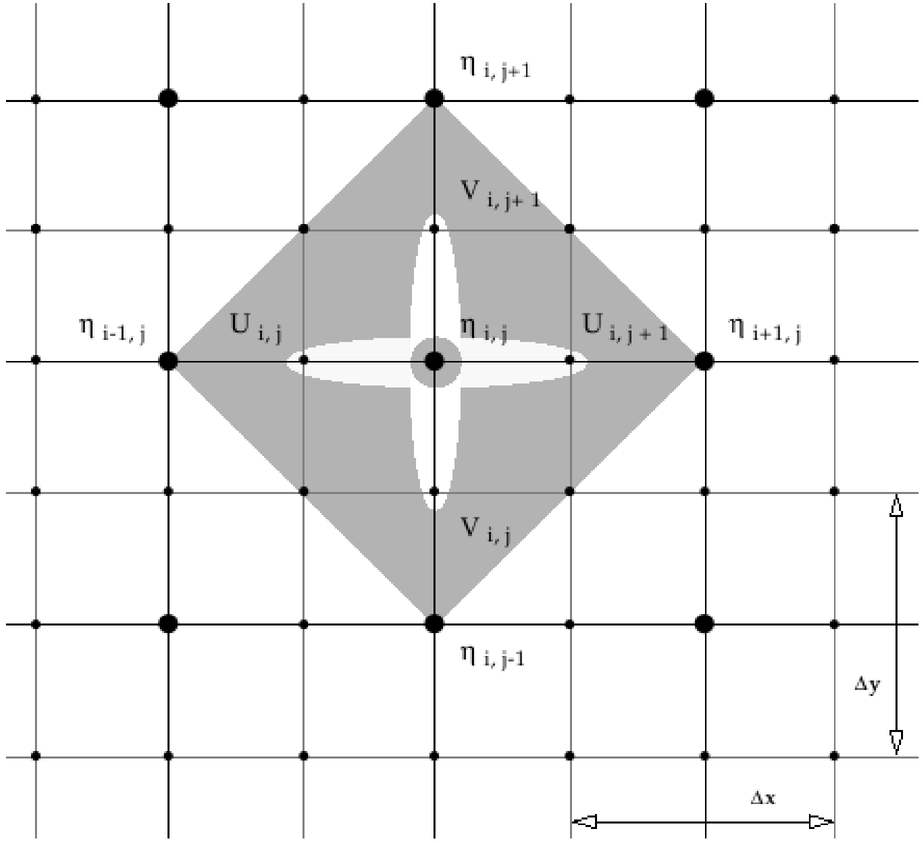
$$+ V_{i,j+1}^{n-1} + V_{i,j-1}^{n-1} - 4V_{i,j}^{n-1})$$

$$+ \quad (\tau_y)_{i,j}^{n} - C_{\rm d}V_{i,j}^{n-1}$$

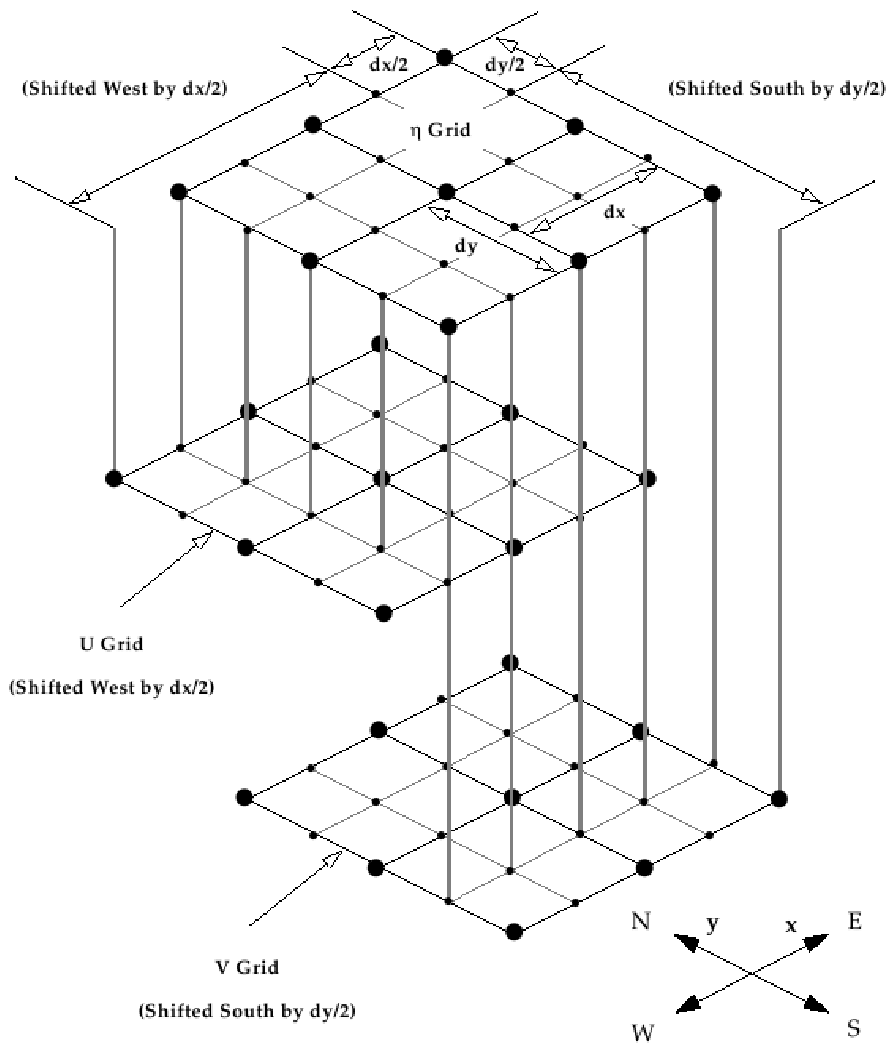
Explicit time discretization

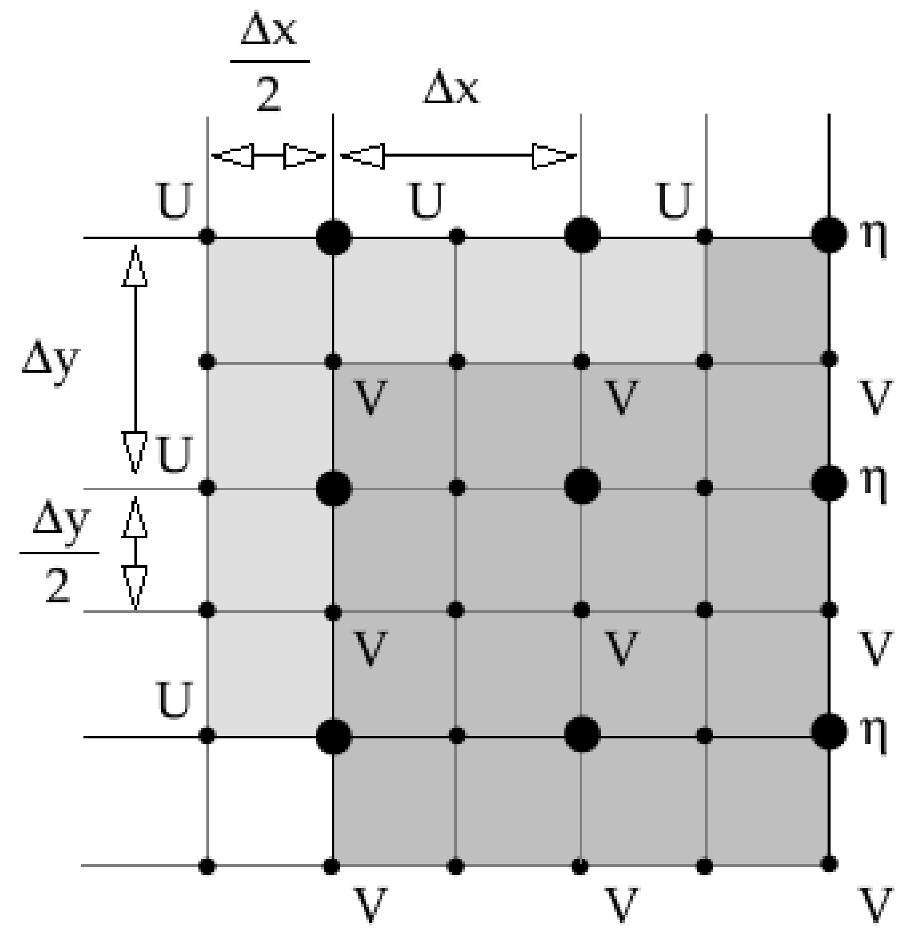
$$\frac{\eta_{i,j}^{n+1} - \eta_{i,j}^{n-1}}{\Delta t}$$

$$= \frac{(U_{i+1,j}^{n} - U_{i,j}^{n})}{\Delta x} - \frac{(V_{i,j+1}^{n} - V_{i,j}^{n})}{\Delta y}$$



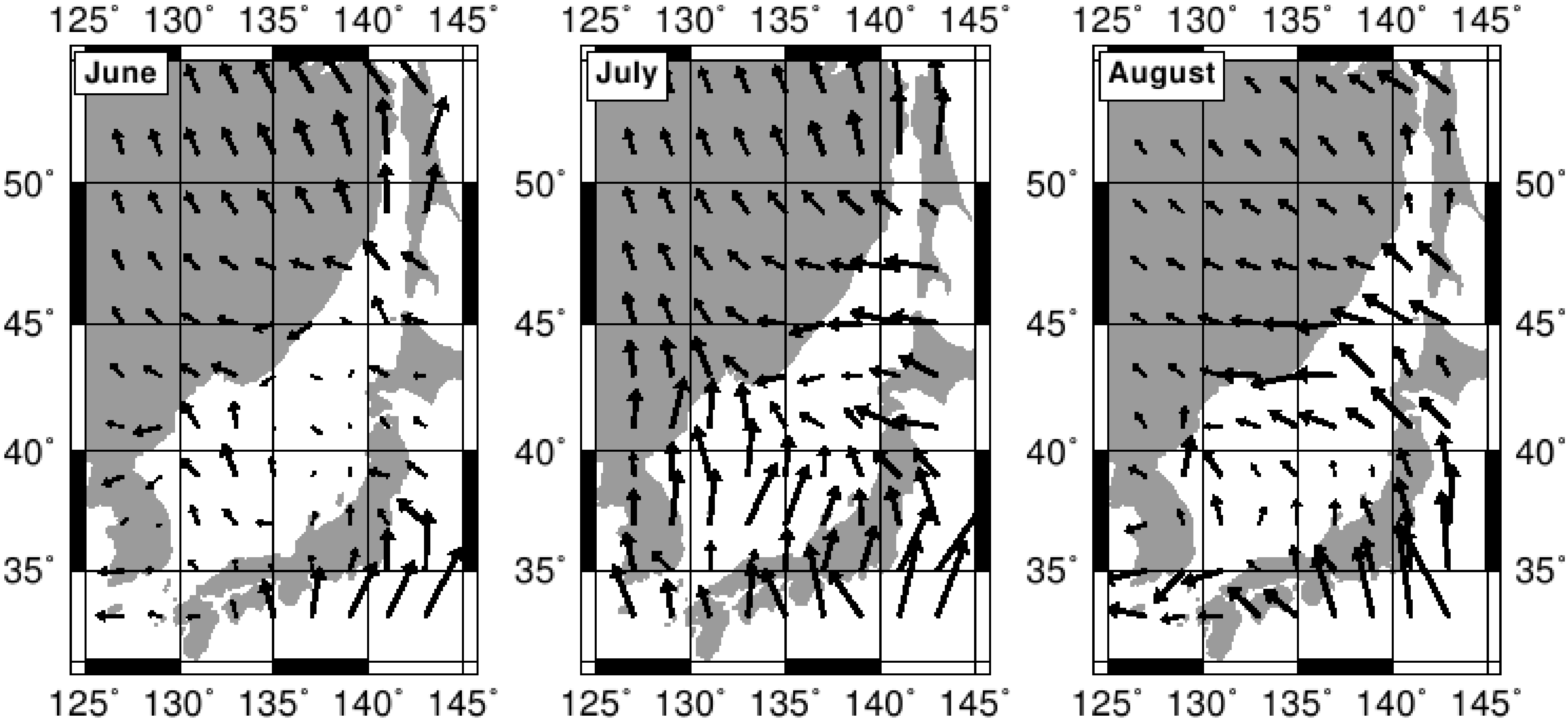
Composite grids





Composite Grid

Forcings

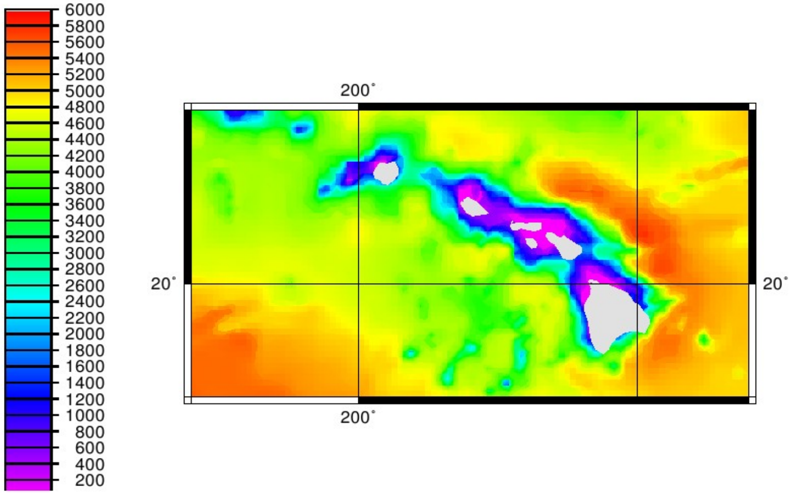


Stability => CFL

Courant-Fiedrichs-Levy (CFL)

$$\Delta t < \Delta x / c_{\mathrm{w}}$$

$$c_{\mathrm{w}} = \sqrt{gH}$$



$\Delta t \approx 100 s$

$\Delta x = 20 km$

H = 4000 m