Instabilities in Geophysical Flows: Barotropic/Baroclinic, Temporal/Spatial

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1 Basic flow

• is an arbitrarily prescribed initial state; not the mean flow itself

• exists under specified external forcing (dissipation, source terms etc)

• is assumed steady, parallel and, furthermore, zonal, i.e.

\[ U_0(x, y, z, t) = U_0(y, z) \]

with \( x \) denoting the streamwise coordinate

• a zonal flow is an inviscid stationary state

(i) \((U_0, P_0)\), with \( P_0 \) constant, solves the incompressible (non–rotational) Euler equations.

(ii) \((U_0, P_0)\) solves the Boussinesq equations

a) without rotation, the basic pressure \( P_0 = P_0(z) \) is in hydrostatic balance with the basic density field \( \bar{\rho} = \bar{\rho}(z) \);

b) with rotation, the basic pressure \( P_0 = P_0(y, z) \) is in geostrophic balance with the basic flow and in hydrostatic balance with the basic density field \( \bar{\rho} = \bar{\rho}(z) \).
(iii) if $U_0 = U_0(y)$, then $(U_0, \eta_0)$ solves the flat bottom (or bottom profile $h_b = h_b(y)$) shallow-water eqs with the height of the undisturbed surface

a) $\eta_0 = H$ constant (and flat bottom) in the absence of rotation;

b) $\eta_0 = \eta_0(y)$ in geostrophic balance with $U_0$ when rotation is present.

(iv) if $U_0 = U_0(y)$, then $\zeta_0 = \zeta_0(y)$ solves the two-dimensional vorticity equation both on the $f$–plane and on the $\beta$–plane. On the $f$–plane any parallel flow is a steady solution; on the $\beta$–plane only a zonal flow solves the unforced equations.

(v) $U_0$ solves the quasigeostrophic potential–vorticity equations with the geostrophic stream function $\Psi$

$$U_0 = -\frac{\partial \Psi}{\partial y}(y,z);$$

on the $\beta$–plane only a zonal flow solves the unforced QG equations; on the $f$–plane any parallel flow is a steady solution
2 Formulation of the stability problem

2.1. Incompressible Euler equations

- consider a perturbation of the basic state
  \[ u(x, y, z, t) = U_0(y, z) \mathbf{i} + \tilde{u}(x, y, z, t), \quad p(x, y, z, t) = P_0 + \tilde{p}(x, y, z, t) \]

- write the eqs for the perturbation fields \((\tilde{u}, \tilde{p})\) and study the evolution of the solution (growth, decay) with respect to arbitrary initial disturbances

- the nonlinear problem is generally intractable and one is forced to consider only small (infinitesimal) perturbations, i.e.

\[
\left( \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} \right) \tilde{u} + \left( \tilde{v} \frac{\partial}{\partial y} + \tilde{w} \frac{\partial}{\partial z} \right) U_0 \mathbf{i} + \nabla \tilde{p} = 0
\]

\[ \nabla \cdot \tilde{u} = 0 \]
• seek solutions in the normal mode form

\[ \tilde{u}(x, y, z, t) = \hat{u}(y, z) e^{i(kx-\omega t)} \]

\[ \tilde{p}(x, y, z, t) = \hat{p}(y, z) e^{i(kx-\omega t)} \]

• typically \( U_0 = U_0(y) \) or \( U_0 = U_0(z) \) – use Squire’s theorem (1933):
  for each unstable 3D disturbance there corresponds a more unstable 2D one

• introduce a perturbation stream function \( \tilde{\psi} \) such that

\[ \tilde{u} = -\frac{\partial \tilde{\psi}}{\partial y}, \quad \tilde{v} = \frac{\partial \tilde{\psi}}{\partial x} \quad \text{or} \quad \tilde{u} = -\frac{\partial \tilde{\psi}}{\partial z}, \quad \tilde{w} = \frac{\partial \tilde{\psi}}{\partial x} \]

and assume that

\[ \tilde{\psi}(x, y, t) = \Phi(y) e^{i(kx-\omega t)} \quad \text{or} \quad \tilde{\psi}(x, z, t) = \Phi(z) e^{i(kx-\omega t)} \]
• consider appropriate boundary conditions in \( y \) (or in \( z \)), eliminate \( \hat{p} \) from the equations and derive an eigenvalue problem where the eigenfunctions \( \Phi(y,k) \) (\( \Phi(z,k) \)) satisfy the Rayleigh’s equation

\[
(U_0 - c) \left( \frac{\partial^2 \Phi}{\partial y^2} - k^2 \Phi \right) - \frac{\partial^2 U_0}{\partial y^2} \Phi = 0, \quad c = \frac{\omega}{k}
\]

and \( k \) and \( \omega \) satisfy a dispersion relation

\[
D[k,\omega] = 0
\]

• temporal modes \( \omega = \omega(k) \) refer to cases where the complex frequency \( \omega \) is determined as a function of the real wave number \( k \)

• writing \( \omega = \omega_r + i \omega_i \), temporal (linear) instability with exponential growth \( e^{\omega_i t} \) occurs if \( \omega_i > 0 \); the temporal growth rate of the wave is \( \omega_i \) and the phase speed \( c_r = \omega_r/k \). If \( \omega_i \leq 0 \), the wave is stable.
Comments:

- the problem for the perturbation \((\tilde{u}, \tilde{p})\) does not depend on external forcing.

- if \((\Phi, c)\) is an eigenfunction/value pair then also \((\phi^*, c^*)\) is one, i.e. to each unstable mode there corresponds a stable mode and stability for this problem means neutral stability \(\omega_i = 0\).

- the Rayleigh's equation has a singularity in the point(s) where the base velocity \(U_0\) equals the phase velocity \(c\); in fact for any given basic flow there exist only a finite number of non-singular normal modes. The spectrum can be completed, so that any smooth initial disturbance can be represented as a superposition of the normal modes, by considering the continuous spectrum of singular normal modes each corresponding to a real value of \(c\) in the range of the basic flow \(U_0\). Note that the continuous spectrum corresponds only to stable modes.
2.2. Continuously stratified quasi–geostrophic equations

- consider a perturbation of the basic state

\[ \psi(x, y, z, t) = \Psi(y, z) + \phi(x, y, z, t) \]

- write the linearized QG eqs for the perturbation stream function \( \phi \)

\[ \left( \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \phi}{\partial z} \right) \right) + \frac{\partial \phi}{\partial x} \frac{\partial \Pi_0}{\partial y} = 0 \]

where \( N^2 = N^2(z) = -\frac{g}{\rho_s} \frac{\partial \rho_s}{\partial z} \) and

\[ \Pi_0 = \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \Psi}{\partial z} \right) + \beta y \]

denotes the potential vorticity of the basic flow.

- in a continuously stratified Boussinesq fluid (oceanic case), \( \rho_s \) may be considered constant, except in the Brunt–Väisälä frequency \( N^2 \).
the zonal basic flow $U_0$ influences the stability problem as itself and through the meridional gradient of its potential vorticity, i.e.

$$\frac{\partial \Pi_0}{\partial y} = \beta - \frac{\partial^2 U_0}{\partial y^2} - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial U_0}{\partial z} \right)$$

since the problem is linear we can seek solutions as superposition of the normal modes at fixed frequency $\omega$ and $x$–wavenumber $k$, i.e.

$$\phi(x, y, z, t) = \text{Re} \Phi(y, z) e^{i(kx-\omega t)}$$

this yields the following PDE (Rayleigh’s equation) for $\Phi$

$$(U_0 - c) \left( \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z} \right) - k^2 \Phi \right) + \Phi \frac{\partial \Pi_0}{\partial y} = 0$$

enforcing appropriate boundary conditions in $y$ and $z$ leads to an eigenvalue problem whereby eigenfunctions $\Phi(y, z; k)$ exist only if $k$ and $\omega$ satisfy a dispersion relation

$$D[k, \omega] = 0$$

the Rayleigh’s equation and the corresponding dispersion relation can often be solved only numerically
3 Barotropic instability

- arises from the horizontal shear in the basic flow
- gravitational and buoyancy effects secondary
- important mechanism for jets and vortices and in turbulence
- can be explained by 2D (barotropic) dynamics

- the driving mechanism is the conversion of kinetic energy from the mean flow to the fluctuations (eddy); situation referred to as down–gradient eddy momentum flux

- recall that a fluid is barotropic when the density is a function of pressure only, i.e. the surfaces of constant pressure and the surfaces of constant density coincide ($\nabla \rho \times \nabla p = 0$)
Example: Stability of free shear layer – Kelvin-Helmholtz instability

- consider a constant–density fluid on the $f$-plane and a zonal basic flow with horizontal shear in the cross-stream direction, i.e.

$$U_0 = U_0(y) = \begin{cases} U, & y > 0 \\ -U, & y < 0 \end{cases}$$

- the perturbation field can be assumed to be two–dimensional and the Rayleigh’s equation simplifies to

$$(U_0 - c) \left( \frac{\partial^2 \Phi_j}{\partial y^2} - k^2 \Phi_j \right) = 0, \quad j = 1, 2$$

with $\Phi_1 = \Phi_1(y)$ ($\Phi_2 = \Phi_2(y)$) denoting the solution in $y > 0$ ($y < 0$).

- considering the boundary conditions $\lim_{y \to \pm \infty} \Phi(y) = 0$ and appropriate continuity (matching) conditions at the interface $y = 0$ we get

$$\Phi(y) = \begin{cases} \Phi_1^0 e^{-ky}, & y > 0 \\ \Phi_2^0 e^{ky}, & y < 0 \end{cases}, \quad \Phi_1^0, \Phi_2^0 \in \mathbb{C}$$

$$-k(U - c) \Phi_1^0 = -k(U + c) \Phi_2^0, \quad \frac{\Phi_1^0}{U - c} = -\frac{\Phi_2^0}{U + c}$$

Hence $c = \pm U i$ and the wave corresponding to $c = U i$ grows exponentially as $e^{kUt}$ when $t \to \infty$
4 Necessary conditions for barotropic instability

• when \( U_0 = U(y)i \) the Rayleigh’s equation can be written as

\[
\Phi_{yy} - k^2 \Phi + \frac{\beta - U_{yy}}{U - c} \Phi = 0
\]

• multiplying by \( \Phi^* \), integrating in \( y \) and assuming that \( \Phi \) vanishes at bdrs

\[
\int (|\Phi_y|^2 + k^2|\Phi|^2) \, dy - \int \frac{\beta - U_{yy}}{U - c} |\Phi|^2 \, dy = 0
\]

• the imaginary part of the equation,

\[
c_i \int \frac{\beta - U_{yy}}{|U - c|^2} |\Phi|^2 \, dy = 0
\]

only if, either \( c_i = 0 \) or the integral vanishes.

• for instability to occur, \( c_i \neq 0 \), hence a necessary condition for instability is that the potential vorticity gradient \( \beta - U_{yy} \) changes sign somewhere in the domain (Kuo’s theorem (1949); when \( \beta = 0 \) it reduces to the Rayleigh’s inflection point criterion (1880))
• Fjørtoft’s (1950) criterium: a necessary condition for instability is that

$$(\beta - U_{yy})(U - U_s)$$

is positive somewhere in the domain. Here $U_s$ is the point where $\beta - U_{yy}$ vanishes.

• note that boundary terms do not play any role in these criteria

• the inflection-point criteria are valid even when the disturbances are not of the normal mode form

• the $\beta-$ effect has often a stabilizing effect but it may destabilize certain velocity profiles, e.g. westward point jet $U(y) = -(1 - |y|)$

• for a linear stability analysis of hyperbolic tangent barotropic shear flow

$$U(y) = \frac{1}{2}(1 + \tanh(y/L))$$

see, e.g., Dickinson and Clare (1973)
**Example:** Stability conditions for a continuously stratified flow.

Consider a zonal basic flow $U_0 = U_0(z)$, non-rotational Boussinesq equations and two-dimensional disturbances in the $xz$–plane. This leads to the Taylor–Goldstein equation

$$(U_0 - c)(\Phi_{zz} - k^2 \Phi) - \frac{\partial^2 U_0}{\partial z^2} \Phi + \frac{N^2}{U_0 - c} \Phi = 0.$$ 

Using the boundary conditions $\Phi(0) = \Phi(H) = 0$, a sufficient condition for linear stability is

$$\text{Ri}(z) = \frac{N^2(z)}{(\partial U_0/\partial z)^2} > \frac{1}{4},$$

where $N^2(z) = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z}$; also called the Richardson number criterion.

• Note that in a homogeneous (constant–density) fluid, the Taylor–Goldstein equation reduces to the Rayleigh’s equation.
Lipps (1963) and Ripa (1983) have studied stability conditions for divergent flows.

- Lipps considered reduced gravity equations (active layer at bottom) and used quasi–geostrophic approximation. He found that a basic flow $U_0 = U_0(y)$ is stable if the meridional gradient of the potential vorticity
  \[
  \frac{\partial \Pi_0}{\partial y} = \left( \beta - \frac{\partial^2 U_0}{\partial y^2} + \frac{f_0^2 L^2}{g r H} U_0 \right)
  \]
does not change sign; $H$ is a constant mean depth of the bottom layer.

- Ripa analysed shallow–water eqs and found sufficient conditions for linear stability of infinitesimal perturbations.
5 Baroclinic instability

- consider a stably stratified fluid in which the isopycnals (constant potential density surfaces) or isentropes (constant potential temperature surfaces) tilt upwards

- this can be a steady state in a rotating fluid since the resulting horizontal pressure gradient can be balanced by the Coriolis force

- if fluid parcels are displaced along a trajectory whose slope $\alpha$ satisfies $0 < \alpha < \phi$ where $\phi$ is the slope of the isopycnals, the buoyancy force will accelerate the fluid (light fluid rises and heavy fluid sinks) and the situation is unstable

- potential energy is released and converted to kinetic energy in a form of thermal convection

- scaling analysis shows that the horizontal length scales most favorable for baroclinic instability are of the order of, or exceed, the Rossby radius of deformation $L_D = \frac{NH}{f_0}$; hence $Bu \leq O(1)$, i.e. fluctuations are of synoptic scale
Linear stability analysis

- the Rayleigh’s equation must be complemented with boundary conditions at the two vertical boundaries, i.e.

\[
(U_0 - c) \frac{\partial \Phi}{\partial z} - \frac{\partial U_0}{\partial z} \Phi = 0 \quad \text{at} \quad z = 0, H
\]

- this equation arises from the buoyancy equation (with \( w = 0 \)) at flat boundaries in the absence of friction

- we hence have the equations

\[
\begin{cases}
(U_0 - c) \left( \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z} \right) - k^2 \Phi \right) + \Phi \frac{\partial \Pi_0}{\partial y} = 0 & y \in (-1, 1), z \in (0, H) \\
\Phi = 0 & y = \pm 1, z \in (0, H) \\
(U_0 - c) \frac{\partial \Phi}{\partial z} - \frac{\partial U_0}{\partial z} \Phi = 0 & y \in (-1, 1), z = 0, H
\end{cases}
\]
6 Necessary conditions for baroclinic instability

- multiply the Rayleigh’s equation by $\rho_s \Phi^*$ and integrate in $y$ and $z$

$$\int_{-1}^{1} \int_{0}^{H} \left( |\Phi_y|^2 + \frac{f_0^2}{N^2} |\Phi_z|^2 + k^2 |\Phi|^2 \right) dy \, dz - \int_{-1}^{1} \int_{0}^{H} \frac{\rho_s |\Phi|^2}{U_0 - c} \frac{\partial \Pi_0}{\partial y} dy \, dz =$$

$$= \int_{-1}^{1} \left[ \frac{f_0^2}{N^2} \frac{\rho_s |\Phi_z|^2}{U_0 - c} \frac{\partial U_0}{\partial z} \right]_{0}^{H} dy$$

- the imaginary part is given by

$$-c_i \int_{-1}^{1} \left\{ \int_{0}^{H} \frac{\rho_s |\Phi|^2}{|U_0 - c|^2} \frac{\partial \Pi_0}{\partial y} dy \, dz + \left[ \frac{f_0^2}{N^2} \frac{\rho_s |\Phi_z|^2}{|U_0 - c|^2} \frac{\partial U_0}{\partial z} \right]_{0}^{H} \right\} dy = 0$$

- for instability $c_i \neq 0$, thus one can divide by $U_0 - c$ since it never vanishes
The Charney–Stern–Pedlosky (Charney–Stern (1962), Pedlosky (1964)) necessary condition for instability:

One of the following criteria must be satisfied:

(i) \( \frac{\partial \Pi_0}{\partial y} \) changes sign in the interior of the domain;

(ii) the sign of \( \frac{\partial \Pi_0}{\partial y} \) is opposite to the sign of \( \frac{\partial U_0}{\partial z} \) at the upper boundary;

(iii) \( \frac{\partial \Pi_0}{\partial y} \) has the same sign as \( \frac{\partial U_0}{\partial z} \) at the lower boundary;

(iv) \( \frac{\partial U_0}{\partial z} \) is of the same sign at upper and lower boundaries; note that this differs from (ii) and (iii) if \( \frac{\partial \Pi_0}{\partial y} = 0 \)

The conditions can be derived without assuming that the disturbances are of the normal–mode form.
7 Eady (1949) problem

(i) uniformly stratified ($N^2$ is constant) Boussinesq ($\rho_s$ is constant) fluid;
(ii) the motion takes place between two rigid, flat surfaces;
(iii) $f$–plane approximation;
(iv) the basic flow is zonal and has uniform vertical shear, i.e.
$$U_0 = U_0(z) = U H^{-1}z, \quad U \in \mathbb{R} \setminus \{0\}$$

- note that the potential–vorticity gradient of the basic flow vanishes
$$\frac{\partial \Pi_0}{\partial y} = \beta - \frac{\partial^2 U_0}{\partial y^2} - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial U_0}{\partial z} \right) = 0$$
Rayleigh’s equations reduce to

\[
\begin{cases}
(UH^{-1}z - c) \left( \frac{\partial^2 \Phi}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \Phi}{\partial z^2} - k^2 \Phi \right) = 0 & \text{if } y \in (-1, 1), \ z \in (0, H) \\
\Phi = 0 & \text{if } y = \pm 1, \ z \in (0, H) \\
-c \frac{\partial \Phi}{\partial z} - UH^{-1} \Phi = 0 & \text{if } y \in (-1, 1), \ z = 0 \\
(U - c) \frac{\partial \Phi}{\partial z} - H^{-1} \Phi = 0 & \text{if } y \in (-1, 1), \ z = H
\end{cases}
\]

• solutions are of the form

\[\Phi(y, z) = \left( A \cosh(\mu H^{-1}z) + B \sinh(\mu H^{-1}z) \right) \cos(ly)\]

where

\[\mu^2 = (k^2 + l^2) L_D^2, \quad L_D = \frac{NH}{f_0}, \quad l = (n + \frac{1}{2}) \pi, \quad n = 0, 1, \ldots\]
• using the boundary conditions, we obtain the dispersion relation

\[ c^2 - Uc + U^2 (\mu^{-1} \coth \mu - \mu^{-2}) = 0 \]

from which it follows that

\[ c = \frac{U}{2} \pm \frac{U}{\mu} \left[ \left( \frac{\mu}{2} - \coth \frac{\mu}{2} \right) \left( \frac{\mu}{2} - \tanh \frac{\mu}{2} \right) \right]^{1/2} \]

• for instability to occur

\[ \frac{\mu}{2} < \coth \frac{\mu}{2} \iff \mu < \mu_c \approx 2.3994 \]

with the corresponding \(x\)-wavenumber and wavelength satisfying

\[ k < k_c = \mu_c \frac{L_D^{-1}}{\mu}, \quad \lambda > \lambda_c = \frac{2\pi L_D}{\mu_c} \approx 2.6 L_D \]

• the growth rates are given by

\[ \sigma = kc_i = k \frac{U}{\mu} \left[ \left( \coth \frac{\mu}{2} - \frac{\mu}{2} \right) \left( \frac{\mu}{2} - \tanh \frac{\mu}{2} \right) \right]^{1/2} \]

and the maximum growth rate occurs at \( \mu = \mu_m \approx 1.61 \); the corresponding non-dimensional growth rate being

\[ \sigma_m = k_m c_i \frac{L_D}{U} \approx 0.31 \]
- the instability arises from the interaction of the waves at the upper and lower boundaries; if one of the boundaries is absent, there is no instability

- there is a short-wave cutoff beyond which instabilities do not occur

- the wavelenght of the most unstable scale is about four times the deformation radius \( L_D \)

- although the Eady problem is quantitatively more appropriate for the atmosphere it gives a qualitative idea of the nature of baroclinic instability in both cases

- the problem meets one of the Charney–Stern–Pedlosky necessary conditions, namely \( \frac{\partial U_0}{\partial z} = U H^{-1} > 0 \) is of the same sign at the upper and lower boundaries
8 Two–layer problem

• consider two fluid layers of equal thicknesses \( H/2 \) and with different densities \( \rho_2 > \rho_1 \) (bottom heavy). Assume uniform basic flows

\[
U^j_0 = (-1)^{j+1} U i \quad j = 1, 2, \quad U > 0
\]

with the corresponding stream functions and potential vorticities

\[
\psi_j(y) = (-1)^j U y, \quad \Pi^j_0(y) = \beta y + (-1)^j \frac{2f_0^2}{g_r H} (\psi_1(y) - \psi_2(y)), \quad j = 1, 2
\]

where \( g_r = g (\rho_2 - \rho_1)/\rho_1 \) is the reduced gravity.

• write the linearized potential vorticity equations for the perturbation stream functions \( \phi_j \) in each layer \( j \), seek normal mode solutions in the form

\[
\phi_j(x, y, t) = \text{Re} \left( \Phi_j e^{i(kx + ly - \omega t)} \right), \quad j = 1, 2
\]

and obtain the dispersion relation

\[
(c K^2 + \beta) (c(K^2 + F^2) + \beta) - U^2 K^2 (K^2 - F^2) = 0
\]

where \( K^2 = k^2 + l^2 \) and \( F^2 = 4f_0^2 (g_r H)^{-1} \).

• solve the dispersion relation for \( c \)

\[
c = -\frac{\beta (2K^2 + F^2)}{2K^2(K^2 + F^2)} \pm \frac{\sqrt{\beta^2 F^4 + 4U^2 K^4 (K^4 - F^4)}}{2K^2(K^2 + F^2)}
\]
• if $U = 0$ we obtain real phase speeds

$$c = -\frac{\beta}{K^2} \quad \text{(barotropic mode)}$$

$$c = -\frac{\beta}{K^2 + F^2} \quad \text{(baroclinic mode)}$$

• if $\beta = 0$ and $U \neq 0$ we obtain

$$c = \pm U \frac{K^2 - F^2}{K^2 + F^2}$$

(i) there is a temporal instability for $KF^{-1} < 1$ (long waves);

(ii) the maximum growth rate occurs when $l = 0$ and $k \approx 0.634 F = 1.79 L_D$;

(iii) there is a high wavenumber cut–off; for $K > F \approx 2.82 L_D^{-1}$ there is no growth, cf. Eady problem.

• in the general case, $\beta, U \neq 0$, an instability occurs if

$$\beta^2 F^4 + 4U^2 K^4 (K^4 - F^4) < 0$$

(i) a minimum shear for instability is $U_s > \beta F^{-2}$ at $K = 2^{-1/4} F$;

(ii) $KF^{-1} < 1$ is a necessary condition for instability (high wavenr cut–off);

(iii) very long waves ($K^{-1} \to \infty$) are stable due to the $\beta$–effect since for instability to occur when $K \ll F$ we require $K^2 > \frac{\beta}{2U}$. 
9 The effect of viscosity

- assume a constant–density, incompressible Navier–Stokes fluid, a zonal basic flow $U_0 = U_0(y)$ and, cf. Squire’s theorem, consider 2D perturbations expressed in terms of a (perturbation) stream function $\phi = \phi(x, y, t)$

- assume that

$$\phi(x, y, t) = \text{Real}\left(\Phi(y)e^{i(kx-\omega t)}\right)$$

- this results in the following fourth–order ODE; Orr–Sommerfeld equation

$$(U_0 - c) (\Phi_{yy} - k^2 \Phi) - U_0,yy \Phi = \frac{1}{i k \text{Re}} \left( \Phi_{yyyy} - 2k^2 \Phi_{yy} + k^4 \Phi \right)$$

where $\text{Re}$ denotes the Reynolds number.

- The no–slip boundary conditions $\tilde{u} = \tilde{v} = 0$ are satisfied if $\Phi = \Phi_y = 0$ at $y = \pm 1$. 
• the Orr–Sommerfeld (1907, 1908) equation, although linear, can in general be approached only asymptotically or numerically.

• for large \( \text{Re} \), when (turbulence) transition is observed to happen, the right-hand side is apparently small containing, however, the highest-order derivatives and leading to regions of rapid variation for the eigenfunctions in which the viscous terms are of the same size as the inertial terms.

• the viscosity can have either stabilizing or destabilizing effect

**Example: Plane Poiseuille Flow**

(i) inviscidly stable (no inflection points);

(ii) linearly viscously unstable at \( \text{Re} = 5772 \); nonlinearly unstable at \( \text{Re} = 2510 \);

(iii) viscosity changes the phase relationship between \( \tilde{u} \) and \( \tilde{v} \) and brings about Reynolds stresses that, by interacting with the shear \( U_{0,y} \), produce disturbance energy that overcomes the viscous dissipation and destabilizes the flow.
Absolute and convective instabilities

- we have been studying, so far, temporal instabilities, i.e. we have been interested in the temporal evolution of an initial disturbance with a (fixed and real) streamwise wavenumber $k$.

- temporal modes $\omega(k) = \omega_r(k) + i\omega_i(k)$ have been obtained as zeros of the dispersion relation $D[k, \omega] = 0$, with the main quantities of interest being the growth rate $\omega_i(k)$ and the phase speed $c_r(k) = \omega_r(k)/k$.

- if there exists a mode and a wavenumber $k$ such that $\omega_i(k) > 0$, the basic flow $U_0$ is linearly unstable; otherwise it is (linearly) stable.

- there are cases, however, where the linear instability appears as spatial growth of a meander pattern as a response to a localized, at $x = 0$ say, initial forcing with a real frequency $\omega$.

(i) if the long–time response decreases with time at any fixed point in space the spatial instability is convective (the instability is advected away) and steady–state solutions exist.

(ii) if the long–time response of the medium grows exponentially in time at any fixed point in space the spatial instability is absolute.
• the spatial modes, or branches, \( k(\omega) = k_r(\omega) + i k_i(\omega) \) are obtained as (possibly complex) zeros of the dispersion relation with real frequency \( \omega \); the spatial growth rate being \( -k_i(\omega) \) and the phase speed \( c_r = \omega / k_r(\omega) \).

• it is not clear, however, whether spatial modes with \( -k_i(\omega) > 0 \) correspond to actually (spatially) growing solutions or are only spurious growing modes.

• think of barotropic Rossby waves in a resting fluid; the dispersion relation \( D[k, \omega] = \omega + \frac{\beta k}{k^2 + l^2} = 0 \) yields

\[
k(\omega) = -\frac{\beta}{2\omega} \pm \left( \frac{\beta^2}{4\omega^2} - l^2 \right)^{1/2}
\]

• we know that the waves are stable; \( \omega \) is real for all real \( k \) and \( l \) with the maximum frequency \( \omega = -\beta/2l \) at \( k = l \).

• suppose that \( -\omega > \beta/2l \), thus

\[
k(\omega) = -\frac{\beta}{2\omega} \pm i \left( l^2 - \frac{\beta^2}{4\omega^2} \right)^{1/2}
\]

and spatially growing solutions, \( -k_i(\omega) > 0 \), exist.

• these are not, however, physically meaningful solutions for \( x > 0 \), they are there since we must be able to represent a forcing at large \( x \), a response to which for smaller \( x \) is decreasing.
• consider a zonal basic flow $U_0 = U_0(y)$ in a homogeneous incompressible NS–fluid

• normal mode approach leads to the Orr–Sommerfeld equation

$$(U_0 - c)(\Phi_{yy} - k^2 \Phi) - U_{0,yy} \Phi = \frac{1}{ik \Re} \left( \Phi_{yyyy} - 2k^2 \Phi_{yy} + k^4 \Phi \right)$$

and enforcement of bdry conditions $\Phi(\pm 1) = \Phi_y(\pm 1)$ to a dispersion relation

$$D[k, \omega, \Re] = 0$$

• Orr–Sommerfeld equation admits non–trivial solutions $\Phi(y)$ (eigenfunctions) if and only if the dispersion relation is satisfied for some complex wavenumber and frequency $(k, \omega)$ (eigenvalue pair).

• associated with the dispersion relation in spectral space is a partial differential operator in physical space so that the fluctuations satisfy

$$D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}, \Re \right) \phi(x, t) = 0$$

• let $G(x, t)$ be the Green’s function of the differential operator, i.e.

$$D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}, \Re \right) G(x, t) = \delta(x) \delta(t)$$

where $\delta$ denotes the Dirac delta function.
(i) the basic flow is linearly stable if

\[ \lim_{t \to \infty} G(x,t) = 0 \quad \text{along all rays} \quad \frac{x}{t} = \text{constant}; \]

(ii) the basic flow is linearly unstable if

\[ \lim_{t \to \infty} G(x,t) = \infty \quad \text{along at least one ray} \quad \frac{x}{t} = \text{constant}; \]

(a) an unstable flow is linearly convective unstable if

\[ \lim_{t \to \infty} G(x,t) = 0 \quad \text{along the ray} \quad \frac{x}{t} = 0; \]

(b) an unstable flow is linearly absolutely unstable if

\[ \lim_{t \to \infty} G(x,t) = \infty \quad \text{along the ray} \quad \frac{x}{t} = 0. \]
• the Green function can be expressed as a double Fourier integral

\[ G(x, t) = \frac{1}{(2\pi)^2} \int_{L_\omega} \int_{F_k} \frac{e^{i(kx-\omega t)}}{D[k, \omega, \text{Re}]} d\omega dk \]

along straight line contours \( L_\omega \) and \( F_k \) in the complex \( \omega- \) and \( k- \) planes.

• \( F_k \) is initially taken along the real axis; \( L_\omega \) is chosen as a horizontal line above all the singularities of the integrand so as to satisfy causality, i.e. \( G(x, t) = 0 \) for all \( x \) when \( t < 0 \).

• assume, for simplicity, that there exists a single temporal mode \( \omega = \omega(k) \); the residue calculation in the \( \omega- \) plane yields formally

\[ G(x, t) = \frac{i}{2\pi} \int_{F_k} \frac{e^{i(kx-\omega(k)t)}}{\frac{\partial D}{\partial \omega}[k, \omega(k), \text{Re}]} dk \]

• consider the behavior of \( G(x, t) \) at large times keeping \( \frac{x}{t} \) fixed (along constant rays); the response is dominated by the real part of the exponent \( \rho = \rho(k) \) defined as

\[ \rho(k) = i\left(k\frac{x}{t} - \omega\right) \]
• to study the long–time dynamics of an integral of the form

\[ G(x, t) = -\frac{i}{2\pi} \int_{F_k} f(k) e^{\rho(k) t} \, dk \]

where \( \rho \) is some (at least) twice differentiable function, we may use the method of steepest descent or saddle–point approximation since, most often, \( \rho \) admits a single stationary point at \( k = k_* \), i.e. \( \frac{\partial \rho}{\partial k}(k_*) = 0 \).

• in the vicinity of \( k_* \), we can write

\[ \rho(k) \approx \rho(k_*) + \frac{1}{2} \frac{\partial^2 \rho}{\partial k^2}(k_*) (k - k_*)^2 \]

thus the surface \( \rho_r(k) \) around \( k_* \) is hyperboloidal and \( k_* \) is a saddle point.

• deforming the original contour \( F_k \) into a steepest descent path \( F_p \) that passes through the saddle point but crosses no other hills so that the global maximum of \( \rho_r \) along \( F_p \) is reached at \( k_* \) and the dominant contribution is expected to arise around \( k_* \).

• using the above approximation for \( \rho \), we are left with a Gaussian integral and by standard arguments finally obtain

\[ G(x, t) \sim \frac{1}{\sqrt{2\pi}} \frac{\exp\left(i\left[\frac{\pi}{4} + k_* x - \omega(k_*) t\right]\right)}{\partial D(\partial \omega_{(k_*)}, \Re\left(\frac{d^2 \omega}{dk^2}(k_*) t\right)\right)^{1/2}} \]
• the temporal growth rate along each ray is
  \[ \sigma = \sigma(V) = \omega_i(k_*) - V k_{*,i} \]
  where \( V \) is the velocity of an observer moving along the ray \( \frac{x}{t} = V \).

• for most problems of interest, the temporal growth rate \( \sigma \) reaches its maximum at \( k_{\text{max}} \) real such that
  \[ \frac{\partial \omega_i}{\partial k}(k_{\text{max}}) = 0 \implies \sigma_{\text{max}} = \omega_i(k_{\text{max}}) \equiv \omega_{\text{max},i} \]
  and the group velocity \( \frac{\partial \omega}{\partial k}(k_{\text{max}}) \equiv V_{\text{max}} \) is real.

• the saddle–point condition implies that
  \[ \frac{x}{t} = \frac{\partial \omega}{\partial k}(k_{\text{max}}) = V_{\text{max}}, \]
  hence, the wave number \( k_{\text{max}} \) is observed along the ray \( \frac{x}{t} = V_{\text{max}} \) where the growth rate attains its global maximum and \( V_{\text{max}} \) provides a measure of the propagation of the centre of the wave packet.

• we conclude that

  (i) if \( \omega_{\text{max},i} > 0 \), the flow is linearly unstable;

  (ii) if \( \omega_{\text{max},i} < 0 \), the flow is linearly stable.
• to distinguish between absolute and convective instabilities, it is necessary to study the wave number \( k_0 \) observed along the ray \( x/t = V = 0 \) which by the saddle point condition satisfies \( \frac{\partial \omega}{\partial k}(k_0) = 0 \).

• the corresponding frequency \( \omega_0 = \omega(k_0) \) is referred to as absolute frequency and the absolute growth rate is \( \sigma(0) = \omega_{0,i} \).

• the absolute growth rate characterizes the temporal evolution of the wave packet as observed at a fixed station; compare with the maximum growth rate \( \omega_{\text{max},i} \) which is observed following the peak of the wave packet.

• the Briggs (1964)–Bers (1983) criterion

(i) if \( \omega_{0,i} > 0 \), the flow is absolutely unstable;

(ii) if \( \omega_{0,i} < 0 \), the flow is convectively unstable.

• the leading– and trailing–edge velocities \( V^+ \) and \( V^- \) are defined as \( \sigma(V^+) = \sigma(V^-) = 0 \), with \( V^+ > V^- \).

• for a convectively unstable flow \( 0 < V^- < V_{\text{max}} < V^+ \); for an absolutely unstable flow \( V^- < 0 < V_{\text{max}} < V^+ \).

• typically, \( \omega_0 \) is an algebraic branch point of \( k(\omega) \) in the complex \( \omega \)–plane or, equivalently, \( k_0 \) is a saddle point of \( \omega(k) \) in the complex \( k \)–plane.
• in wake flows, local absolute instability can lead to global linear instability and to subsequent self–excited nonlinear states, cf. Koch (1985).

• Monkewitz (1988) studied a two–dimensional cylinder wake and showed, as the Reynolds number $Re$ is raised, the sequence of transitions

(i) stability $\Rightarrow$ local convective instability at $Re \approx 5$;

(ii) local convective instability $\Rightarrow$ local absolute instability $Re \approx 25$;

(iii) bifurcation to a self–sustained global mode (linear global instability).

• note that the onset of von Kármán vortex shedding has been experimentally observed at $Re \approx 47$.

• Chen and Jirka (1997) studied absolute and convective instabilities in plane shallow wake flows and obtained critical values for the shallow wake parameter $S = c_f D/H$ where $c_f$ is the coefficient of a quadratic bottom friction–law, $D$ is the island diameter and $H$ the water depth, see also Kolyshkin and Ghidaoui (2003).
References


